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TOPICAL REVIEW

Analytic representations in quantum mechanics

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Abstract

Various Euclidean, hyperbolic and elliptic analytic representations are introduced and relations among them are discussed. The Bargmann analytic representation in the complex plane is considered and its relation to other phase-space methods for the harmonic oscillator is reviewed. The general theory that relates the growth of analytic functions with the density of their zeros is applied to Bargmann functions and it leads to theorems on the completeness of sequences of Glauber coherent states. Two hyperbolic analytic representations in the unit disc, based on $SU(1,1)$ coherent states and also on phase states are introduced. A third analytic representation in the complex plane based on Barut–Girardello states is also considered and transformations which relate it to the other ones are studied. In the case of systems with finite-dimensional Hilbert space, an elliptic analytic representation in the extended complex plane and also another analytic representation based on theta functions are introduced. The Berezin formalism in the Euclidean, hyperbolic and elliptic cases is discussed. Contour analytic representations in these three cases are also presented.

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Contents

1. Introduction

Analytic functions play an important role in various branches of physics. They use the powerful theory of complex analysis to derive physical results. In this review, we discuss analytic representations in the context of coherent states, phase-space methods and their applications. These techniques provide a powerful toolbox in areas such as quantum optics, quantum information processing, condensed matter, atom–light interactions, Bose–Einstein condensates, quantum maps and quantum chaos, condensed matter, mathematical physics, etc.

Coherent states [1–14] have many interesting properties. In this review, we are interested in the fact that they are labelled with a complex variable z and that there is a resolution of the identity in terms of them. We then represent an arbitrary state $|f\rangle$ with an analytic function $f(z)$ which is proportional to the overlap between the state $|f\rangle$ and the coherent states. The resolution of the identity is used to define the scalar product of two states. There is a series of theorems that impose constraints on the behaviour of analytic functions. For example, let $\{z_N\}$ be a sequence of complex numbers which has a limit ζ. Two entire functions $f(z)$ and $g(z)$ such that $f(z_N) = g(z_N)$ for all N are identical. This behaviour of analytic functions can be used to prove the fact that the corresponding sequence of coherent states is overcomplete.

More generally, some of the deeper properties of coherent states are intimately related to analyticity.

Many types of coherent states are associated with a group of transformations. Under these transformations, a coherent state evolves into another coherent state. From a physical point of view, a system with a Hamiltonian which is a linear combination of the generators of this group, and which at $t = 0$ is in a coherent state, will evolve into other coherent states at later times. This property has been called 'temporal stability' [15]. We consider Euclidean analytic representations in the complex plane related to the Heisenberg–Weyl group of displacements, hyperbolic analytic representations in the unit disc related to the $SU(1, 1)$ group [16–19] and elliptic analytic representations in the extended complex plane related to the $SU(2)$ group of rotations [19–22].

The Euclidean analytic representations are related to Glauber coherent states and displacements in the Euclidean plane. We first discuss displacement operators, the Heisenberg–Weyl group and the displaced parity operators. They are intimately related to the Weyl and Wigner functions [23]. Since the Weyl and Wigner functions have been reviewed extensively in the literature [24–27], we only discuss certain aspects which are related to the analytic representations. We then introduce the Bargmann analytic representation in the complex plane [28, 29] and study its properties. We also discuss briefly the Berezin formalism [30] which is an autonomous approach to quantization that is particularly useful in studies of the semiclassical limit.

The growth of a Bargmann function is related to the density of its zeros [31]. Physically, the zeros of an analytic function representing a particular state indicate which coherent states are orthogonal to this state. We show that if the density of a set of complex numbers $\{z_N\}$ is above a certain value then the set of the corresponding coherent states is overcomplete, and if it is below that value it is undercomplete. In the latter case, we can construct a state which is orthogonal to all coherent states $\{|z_N\rangle_{\mathrm{c}}\}$, using Hadamard's theorem which is discussed here in a physical language. These general results are applied to the von Neumann lattice of coherent states [32–38] comprised of the coherent states $\{z_{MN} = S^{1/2}(M + \mathrm{i}N)\}$ where M, N are integers and S is the area of the lattice cell. The zeros of Bargmann functions (and also of other similar representations of quantum states) are important in studies of quantum maps and quantum chaos [39–44].

The Euclidean contour representation assigns to the ket states the same analytic function as the Bargmann representation and to the bra states a function such that the scalar product is given by a contour integral around the origin. It has been studied originally by Dirac [45] and Schwinger [46], and later by several authors [47–51]. In some cases, there are convergence difficulties in this formalism, which we discuss.

We next consider hyperbolic analytic representations. In order to introduce them, we first discuss briefly $SU(1, 1)$ coherent states and the Barut–Girardello states [52, 53]. We also discuss the $k = 1/4$ and $k = 3/4$ representations and their use in the harmonic oscillator formalism, the $k = 1/2$ representation and its use in the description of phase states of the harmonic oscillator, and the Schwinger representation of $SU(1, 1)$ and its use in two-mode harmonic oscillators.

The first hyperbolic analytic representation in the unit disc is based on $SU(1, 1)$ coherent states. It uses functions in the Bergman space [54] and has been studied extensively in the literature [55–64]. We define this representation and explain that $SU(1, 1)$ transformations are implemented with Möbius conformal mappings. We also discuss the Berezin formalism in this context.

Another hyperbolic analytic representation in the unit disc is based on $SU(1, 1)$ phase states. It uses functions in the Hardy space [65] and is related to $\mathcal{Z}$-transform which is used extensively in digital signal processing [66]. The properties of functions in Hardy spaces have been studied extensively in a pure mathematics context and here we use them in a more physical context. For example, we explain that the outer parts of these functions are related to the phase distribution of the corresponding quantum states. The inner parts of these functions contain all the zeros, and theorems about these zeros lead to criteria about the overcompleteness of the phase states.

The Barut–Girardello analytic representation is defined in the complex plane. We discuss this formalism and we give transformations which relate it to the hyperbolic analytic representation in the unit disc based on $SU(1,1)$ coherent states. We also discuss applications to the harmonic oscillator and show that in a certain cases the Barut–Girardello analytic representation is related to the Bargmann representation.

The hyperbolic contour representation is based on a similar philosophy with the Euclidean contour representation. It assigns to the ket states the same analytic function as the hyperbolic analytic representation based on $SU(1,1)$ coherent states and to the bra states a function such that the scalar product is given by a contour integral around the origin. As in the Euclidean case, there are convergence difficulties in this formalism, which we discuss.

In the elliptic case, we consider systems with finite-dimensional Hilbert space. The general theory for such systems has been originally discussed by Weyl [67] and Schwinger [68] and has been recently reviewed in [69]. Here we only discuss analytic representations of these systems.

We first study briefly $SU(2)$ coherent states and their use in the harmonic oscillator formalism. We then define the elliptic analytic representation in the extended complex plane and explain that $SU(2)$ transformations are implemented with Möbius conformal mappings. We also discuss the Berezin formalism in this context. The elliptic contour representation is similar to the Euclidean and hyperbolic contour representations, but all sums are finite here and there are no convergence difficulties.

Another important analytic representation of systems with finite-dimensional Hilbert space on a torus is based on theta functions. This representation provides an interesting link between the theory of systems with finite-dimensional Hilbert space and the theory of theta functions.

Other analytic representations in related contexts have been studied in [70–90]. Path integrals with coherent states using analytic methods have been studied in [91]. More specialized types of coherent states [92] can also be approached in the language of analytic representations [93]. In a different context, analytic methods have also been used in conformal field theory [94] and also in other areas of mathematical physics [95].

The structure of this review aims to show the similarities between the Euclidean, hyperbolic and elliptic cases. Proofs of the formulae are omitted and only hints are given.

2. Notation

$C, R, \mathcal{Z}$ are the sets of complex numbers, real numbers and integers, respectively. $\mathcal{Z}_d$ is the set of integers modulo d. The indices 'R' and 'I' in complex numbers indicate the real and imaginary parts, respectively. The notation ℓ indicates anticlockwise contours in contour integrals. $\mathcal{S}(r)$ and $\mathcal{T}(r)$ indicate the following regions in the complex plane:

$$\mathcal{S}(r) = \{|z| < r\}, \quad \mathcal{T}(r) = \{|z| > r\}. \tag{1}$$

Most states are labelled with an index which indicates the nature of the state. For example, 'c' stands for coherent states, 'n' stands for number eigenstates, 'x' stands for position eigenstates, 'p' stands for momentum eigenstates, 'ph' stands for phase states, etc.

In this review, we work in Euclidean, hyperbolic (Lobachevsky) and elliptic geometry [96] and we give briefly below the metric, the Poisson bracket and the Laplace–Beltrami operator for each of them.

2.1. Euclidean geometry

In the first part of this review, we work in Euclidean geometry in the complex plane C. We use the index 'E' which stands for Euclidean in the various quantities in this case. The metric is given by

$$\mathrm{d}\mu_{\mathrm{E}}(z) = \frac{\mathrm{d}z_{\mathrm{R}}\,\mathrm{d}z_{\mathrm{I}}}{\pi}. \tag{2}$$

The Poisson bracket is

$$\{f, g\}_{\mathrm{E}} = \mathrm{i}\frac{\partial(f, g)}{\partial(z^*, z)} = \frac{1}{2}\frac{\partial(f, g)}{\partial(z_{\mathrm{R}}, z_{\mathrm{I}})}, \tag{3}$$

where

$$\frac{\partial(f, g)}{\partial(\alpha, \beta)} \equiv \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial \beta} - \frac{\partial g}{\partial \alpha}\frac{\partial f}{\partial \beta}. \tag{4}$$

The Laplace operator is

$$\Delta_z^{(\mathrm{E})} \equiv 4\frac{\partial^2}{\partial z \partial z^*} = \frac{\partial^2}{\partial z_{\mathrm{R}}^2} + \frac{\partial^2}{\partial z_{\mathrm{I}}^2}. \tag{5}$$

2.2. Hyperbolic geometry

In the second part of this review, we work in hyperbolic (Lobachevsky) geometry. We use the index 'H' which stands for hyperbolic in the various quantities in this case. We use the Poincare model of the Lobachevsky geometry which is the unit disc

$$D = \{|z| < 1\} = \mathcal{S}(1). \tag{6}$$

In this case, the metric is

$$\mathrm{d}\mu_{\mathrm{H}}(z) = \frac{\mathrm{d}z_{\mathrm{R}}\,\mathrm{d}z_{\mathrm{I}}}{(1 - |z|^2)^2}. \tag{7}$$

The hyperbolic Poisson bracket is given by

$$\{f, g\}_{\mathrm{H}} = \mathrm{i}(1 - |z|^2)^2\frac{\partial(f, g)}{\partial(z^*, z)} \tag{8}$$

and the hyperbolic Laplace–Beltrami operator is given by

$$\Delta_z^{(\mathrm{H})} \equiv (1 - |z|^2)^2\frac{\partial^2}{\partial z \partial z^*} \tag{9}$$

The Laplace–Beltrami operator is invariant under Möbius transformations in the unit disc, which are discussed later in equation (247). Here we explain briefly this invariance. Given a function $g(z)$, we first perform the Möbius transform $w(z)$ of equation (247) to get the function $(g \circ w)(z)$, and then apply the Laplace–Beltrami operator to get $\Delta_z^{(\mathrm{H})}(g \circ w)(z)$. Alternatively, we first apply the Laplace–Beltrami operator to get $\big(\Delta_z^{(\mathrm{H})} g\big)(z)$ and then perform the Möbius transform $w(z)$ to get $\big(\Delta_z^{(\mathrm{H})} g\big)(w(z))$. We then have

$$\Delta_z^{(\mathrm{H})}(g \circ w)(z) = \big(\Delta_z^{(\mathrm{H})} g\big)(w(z)). \tag{10}$$

2.3. Elliptic geometry

In the third part of this review, we work in elliptic geometry. We use the index 'S' which stands for spherical in the various quantities in this case. We consider a sphere where a point is described in spherical coordinates with the angles (α, β), where $0 \leqslant \alpha \leqslant \pi, 0 \leqslant \beta < 2\pi$. A sphere is topologically equivalent to the extended complex plane $C_{\mathrm{E}} = C \cup \{\infty\}$ and the stereographic projection

$$z = -\tan\left(\frac{\alpha}{2}\right) \mathrm{e}^{-\mathrm{i}\beta} \tag{11}$$

provides a one-to-one mapping between the two. The south pole is mapped to the point $z = 0$ and the north pole to ∞. The metric in the extended complex plane is

$$\mathrm{d}\mu_{\mathrm{S}}(z) = \frac{\mathrm{d}z_{\mathrm{R}}\,\mathrm{d}z_{\mathrm{I}}}{(1 + |z|^2)^2}. \tag{12}$$

The elliptic Poisson bracket is given by

$$\{f, g\}_{\mathrm{S}} = \mathrm{i}(1 + |z|^2)^2 \frac{\partial(f, g)}{\partial(z^*, z)} \tag{13}$$

and the elliptic Laplace–Beltrami operator is given by

$$\Delta_z^{(\mathrm{S})} \equiv (1 + |z|^2)^2 \frac{\partial^2}{\partial z \partial z^*}. \tag{14}$$

The Laplace–Beltrami operator is invariant under the Möbius transformations $w(z)$ in the extended complex plane, which are discussed later in equation (396). Given a function $g(z)$, we get

$$\Delta_z^{(\mathrm{S})}(g \circ w)(z) = \big(\Delta_z^{(\mathrm{S})} g\big)(w(z)). \tag{15}$$

Part I: Euclidean analytic representations

3. Harmonic oscillator formalism

3.1. Creation and annihilation operators and their polar decomposition

We consider a harmonic oscillator with states in a Hilbert space $\mathcal{H}$. Let x, p be the position and momentum operators and $a^\dagger, a$ the creation and annihilation operators:

$$\begin{aligned} &a = 2^{-1/2}[x + \mathrm{i}p], \qquad &&a^\dagger = 2^{-1/2}[x - \mathrm{i}p], \\ &[x, p] = \mathrm{i}\mathbf{1}, &&[a, a^\dagger] = \mathbf{1}. \end{aligned} \tag{16}$$

We use the notation $|q\rangle_x$ and $|q\rangle_p$ for the position and momentum eigenstates, and the notation $f_x(q) = {}_x\langle q|f\rangle$ and $f_p(q) = {}_p\langle q|f\rangle$ for the wavefunctions of a state $|f\rangle$ in the position and momentum representations, respectively. They are related through the Fourier transform

$$f_p(r) = (2\pi)^{-1/2} \int \mathrm{d}q\, f_x(q)\, \mathrm{e}^{-\mathrm{i}qr}, \qquad {}_x\langle q|r\rangle_p = (2\pi)^{-1/2}\, \mathrm{e}^{\mathrm{i}qr}. \tag{17}$$

We introduce the projection operators

$$\Pi_x(q) = |q\rangle_{x\,x}\langle q|, \qquad \Pi_p(q) = |q\rangle_{p\,p}\langle q|. \tag{18}$$

The eigenstates of the number operator $n = a^\dagger a$ are

$$|N\rangle_n = \frac{(a^\dagger)^N}{(N!)^{1/2}} |0\rangle_n, \qquad n|N\rangle_n = N|N\rangle_n, \tag{19}$$

where the index 'n' indicates number states.

We introduce a polar decomposition of the 'Cartesian operators' $a^\dagger, a$:

$$a = (n+1)^{1/2}E_- = E_- n^{1/2}, \qquad a^\dagger = E_+(n+1)^{1/2} = n^{1/2}E_+,$$
$$E_+ = E_-^\dagger = \sum_{N=0}^{\infty} |N+1\rangle_{n\,n}\langle N|. \tag{20}$$

E_- and E_+ are 'harmonic oscillator exponential of the phase' operators. The eigenstates of E_- are called phase states and we discuss their properties later in connection with the analytic techniques in the unit disc. In the mathematical literature, E_+ is known as the shift operator.

We next prove that

$$E_-E_+ = \mathbf{1}, \qquad E_+E_- = \mathbf{1} - |0\rangle_{n\,n}\langle 0|. \tag{21}$$

It is seen that E_+ is an isometric operator, but it is *not* a unitary operator. A unitary transformation maps a Hilbert space onto itself. An isometric but non-unitary transformation maps a Hilbert space onto one of its subspaces. In this case, E_+ maps $\mathcal{H}$ onto $\mathcal{H} - \{|0\rangle_n\}$.

3.2. Fractional Fourier operators

We define the Fourier operator as

$$F = \exp\left(\mathrm{i}\frac{\pi}{2}n\right), \qquad F^4 = \mathbf{1} \tag{22}$$

and we can show that

$$F|q\rangle_x = |q\rangle_p, \qquad FxF^\dagger = p, \qquad FpF^\dagger = -x. \tag{23}$$

A more general operator is the fractional Fourier operator [97–100]

$$V(\theta) = \exp(\mathrm{i}\theta a^\dagger a), \qquad V\left(\frac{\pi}{2}\right) = F, \tag{24}$$

which performs rotations in the *x*–*p* phase space:

$$V(\theta)xV(\theta)^\dagger = x\cos\theta + p\sin\theta,$$
$$V(\theta)pV(\theta)^\dagger = -x\sin\theta + p\cos\theta. \tag{25}$$

For later use, we introduce the eigenstates of the operator $V(\theta)xV(\theta)^\dagger$:

$$|q\rangle_\theta \equiv V(\theta)|q\rangle_x. \tag{26}$$

They are position states along the rotated axis $x\sin\theta - p\cos\theta = 0$ in the *x*–*p* phase space. Clearly, $|q\rangle_0 = |q\rangle_x$ and $|q\rangle_{\pi/2} = |q\rangle_p$. We also introduce the projection operators

$$\Pi_\theta(q) = |q\rangle_{\theta\,\theta}\langle q| = V(\theta)\Pi_x(q)V(\theta)^\dagger. \tag{27}$$

We use the notation $f_\theta(q) = {}_\theta\langle q|f\rangle$ for the wavefunction in the 'x_θ representation' (with $f_0(q) = f_x(q)$ and $f_{\pi/2}(q) = f_p(q)$). It is known that

$$f_\theta(q) = \int \mathrm{d}r\, f_x(r)\Delta(q, r; -\theta), \tag{28}$$

where

$$\Delta(q, r; \theta) = \left[\frac{1 + \mathrm{i}\cot\theta}{2\pi}\right]^{1/2} \exp\left[-\mathrm{i}\frac{q^2 + r^2}{2\tan\theta} + \mathrm{i}\frac{qr}{\sin\theta}\right]. \tag{29}$$

An important special case of the fractional Fourier operator is for $\theta = \pi$. In this case, we get the parity operator (around the origin) P_0:

$$P_0 \equiv V(\pi) = \exp(\mathrm{i}\pi a^\dagger a), \qquad P_0^2 = \mathbf{1}, \qquad P_0|q\rangle_x = |-q\rangle_x,$$
$$P_0|q\rangle_p = |-q\rangle_p, \qquad P_0|N\rangle_n = (-1)^N|N\rangle_n. \tag{30}$$

We call $\mathcal{H}_{\text{even}}$ the subspace of the Hilbert space $\mathcal{H}$ spanned by the even number states $|2N\rangle_n$, and $\mathcal{H}_{\text{odd}}$ the subspace spanned by the odd number states $|2N+1\rangle_n$. We also call Π_{even} and Π_{odd} the projection operators to these subspaces, respectively. Then

$$P_0 = \Pi_{\text{even}} - \Pi_{\text{odd}}, \qquad \Pi_{\text{even}} + \Pi_{\text{odd}} = \mathbf{1}. \tag{31}$$

3.3. Displacement operators

In the x–p phase space of the harmonic oscillator, we define displacement operators:

$$D(z) = \exp[za^\dagger - z^* a], \tag{32}$$

where z is a complex number. The product of two such operators is given by

$$\begin{aligned} D(z_1)D(z_2) &= D(z_1+z_2)\exp\left[\tfrac{1}{2}(z_1 z_2^* - z_1^* z_2)\right] \\ &= D(z_2)D(z_1)\exp(z_1 z_2^* - z_1^* z_2). \end{aligned} \tag{33}$$

If we consider products of the displacement operators with phase factors

$$d(z,\gamma) \equiv \mathrm{e}^{\mathrm{i}\gamma} D(z), \qquad d(z_1,\gamma_1)d(z_2,\gamma_2) = d\big(z_1+z_2, \gamma_1+\gamma_2+\tfrac{1}{2}(z_1 z_2^* - z_1^* z_2)\big), \tag{34}$$

we can show that they form a representation of the Heisenberg–Weyl group.

The matrix elements of the displacement operators with respect to positions and momentum eigenstates are

$$\begin{aligned} {}_x\langle q_1|D(z)|q_2\rangle_x &= \exp[\mathrm{i}2^{-1/2}z_{\mathrm{I}}(q_1+q_2)]\delta(q_1-q_2-2^{1/2}z_{\mathrm{R}}), \\ {}_p\langle q_1|D(z)|q_2\rangle_p &= \exp[-\mathrm{i}2^{-1/2}z_{\mathrm{R}}(q_1+q_2)]\delta(q_1-q_2-2^{1/2}z_{\mathrm{I}}), \end{aligned} \tag{35}$$

and with respect to number eigenstates [101, 102] are

$${}_n\langle M|D(z)|N\rangle_n = \exp\left(-\frac{1}{2}|z|^2\right)\left(\frac{N!}{M!}\right)^{1/2} L_N^{M-N}(|z|^2) z^{M-N}, \tag{36}$$

where $L_N^{M-N}(x)$ are associated Laguerre polynomials, defined as in [103].

We can prove the following 'marginal properties' [104]:

$$\begin{aligned} &\frac{1}{2^{1/2}\pi}\int \mathrm{d}z_{\mathrm{R}} D(z) = P_0 \Pi_p(-2^{-1/2}z_{\mathrm{I}}), \\ &\frac{1}{2^{1/2}\pi}\int \mathrm{d}z_{\mathrm{I}} D(z) = P_0 \Pi_x(-2^{-1/2}z_{\mathrm{R}}), \\ &\frac{1}{2}\int_C \mathrm{d}\mu_{\mathrm{E}}(z) D(z) = P_0. \end{aligned} \tag{37}$$

They are used below to prove analogous marginal properties for the Weyl functions.

Another important property of the displacement operators is the 'generalized resolution of the identity'. Let $\mathcal{U}$ be an arbitrary trace class operator acting on $\mathcal{H}$. We prove that

$$\int_C \mathrm{d}\mu_{\mathrm{E}}(z) D(z) \frac{\mathcal{U}}{\operatorname{Tr}\mathcal{U}} [D(z)]^\dagger = \mathbf{1}. \tag{38}$$

There are various ways to prove this. An easy way is to use the P representation of the operator which is discussed later. Acting with the displacement operators $D(z)$ and $[D(z)]^\dagger$ on both sides of the operator $\mathcal{U}$ in equation (78), we can prove equation (38). Although this particular proof uses quantities which are introduced later, logically this relation is a property of displacement operators and belongs in this section. In the special case that $\mathcal{U} = |0\rangle_{n\,n}\langle 0|$, this relation becomes the resolution of the identity for coherent states which is discussed later.

3.4. Weyl functions

The displacement operators are intimately related to the Weyl functions. We define the Weyl function of an operator $\mathcal{U}$ as

$$\begin{aligned}\tilde{W}_{\rm E}(z;\mathcal{U}) &= {\rm Tr}[\mathcal{U}D(z)]\\ &= \int {\rm d}q\, \mathcal{U}_x(q-2^{-1/2}z_{\rm R}, q+2^{-1/2}z_{\rm R})\exp({\rm i}2^{1/2}z_{\rm I}q)\\ &= \int {\rm d}q\, \mathcal{U}_p(q-2^{-1/2}z_{\rm I}, q+2^{-1/2}z_{\rm I})\exp(-{\rm i}2^{1/2}z_{\rm R}q),\end{aligned} \tag{39}$$

where $\mathcal{U}_x(q,r)$ and $\mathcal{U}_p(q,r)$ are the matrix elements

$$\mathcal{U}_x(q,r) \equiv {}_x\langle q|\mathcal{U}|r\rangle_x, \qquad \mathcal{U}_p(q,r) \equiv {}_p\langle q|\mathcal{U}|r\rangle_p. \tag{40}$$

The tilde in the notation reflects the fact that the Weyl function is the two-dimensional Fourier transform of the Wigner function, as we will explain later.

The operator $\mathcal{U}$ can be expanded in terms of the displacement operators with the Weyl functions as coefficients:

$$\mathcal{U} = \int {\rm d}\mu_{\rm E}(z)\tilde{W}_{\rm E}(-z;\mathcal{U})D(z). \tag{41}$$

The marginal properties of the displacement operators in equation (37) lead straightforwardly to the following marginal properties for the Weyl functions:

$$\begin{aligned}&\frac{1}{2^{1/2}\pi}\int {\rm d}z_{\rm R}\, \tilde{W}_{\rm E}(z;\mathcal{U}) = \mathcal{U}_p(-2^{-1/2}z_{\rm I}, 2^{-1/2}z_{\rm I}),\\ &\frac{1}{2^{1/2}\pi}\int {\rm d}z_{\rm I}\, \tilde{W}_{\rm E}(z;\mathcal{U}) = \mathcal{U}_x(-2^{-1/2}z_{\rm R}, 2^{-1/2}z_{\rm R}),\\ &\frac{1}{2}\int_C {\rm d}\mu_{\rm E}(z)\, \tilde{W}_{\rm E}(z;\mathcal{U}) = {\rm Tr}[\mathcal{U}P_0].\end{aligned} \tag{42}$$

There are more marginal properties for the Weyl functions, which involve $|\tilde{W}_{\rm E}(z;\mathcal{U})|^2$ [105]:

$$\begin{aligned}&\frac{1}{2^{1/2}\pi}\int {\rm d}z_{\rm R}|\tilde{W}_{\rm E}(z;\mathcal{U})|^2 = \int {\rm d}q|\mathcal{U}_p(q, q+2^{1/2}z_{\rm I})|^2,\\ &\frac{1}{2^{1/2}\pi}\int {\rm d}z_{\rm I}|\tilde{W}_{\rm E}(z;\mathcal{U})|^2 = \int {\rm d}q|\mathcal{U}_x(q, q+2^{1/2}z_{\rm R})|^2,\\ &\int_C {\rm d}\mu_{\rm E}(z)|\tilde{W}_{\rm E}(z;\mathcal{U})|^2 = {\rm Tr}[\mathcal{U}^\dagger\mathcal{U}].\end{aligned} \tag{43}$$

3.5. Displaced parity operators

The displaced parity operators are defined as

$$P(z) = D(z)P_0[D(z)]^\dagger = D(2z)P_0 = P_0[D(2z)]^\dagger. \tag{44}$$

They perform parity transformations around the point z. The product of two such operators is

$$P(z_1)P(z_2) = \exp[2(z_1^*z_2 - z_1z_2^*)]D(2z_1 - 2z_2). \tag{45}$$

The matrix elements of the displaced parity operators with respect to positions and momentum eigenstates are

$$\begin{aligned}{}_x\langle q_1|P(z)|q_2\rangle_x &= \exp[{\rm i}z_{\rm I}(4z_{\rm R} - 2^{3/2}q_2)]\delta(q_1+q_2-2^{3/2}z_{\rm R}),\\ {}_p\langle q_1|P(z)|q_2\rangle_p &= \exp[-{\rm i}z_{\rm R}(4z_{\rm I} - 2^{3/2}q_2)]\delta(q_1+q_2-2^{3/2}z_{\rm I}).\end{aligned} \tag{46}$$

The displaced parity operators are related to the displacement operators through a two-dimensional Fourier transform:

$$P(\zeta) = \frac{1}{2}\int \mathrm{d}\mu_{\mathrm{E}}(z) D(z) \exp[2\mathrm{i}(\zeta_{\mathrm{I}} z_{\mathrm{R}} - \zeta_{\mathrm{R}} z_{\mathrm{I}})]. \tag{47}$$

The conjugate variables in this two-dimensional Fourier transform are $(z_{\mathrm{R}}, \zeta_{\mathrm{I}})$ and also $(z_{\mathrm{I}}, \zeta_{\mathrm{R}})$. Equation (47) is easily proved if we multiply the two sides of the third of equations (37) by $D(\zeta)$ and $[D(\zeta)]^{\dagger}$.

The marginal properties for the displacement operators given in equation (37) lead to the following marginal properties for the displaced parity operators [104]:

$$\begin{aligned}
&\frac{2^{1/2}}{\pi}\int \mathrm{d}z_{\mathrm{R}} P(z) = \Pi_p(2^{1/2} z_{\mathrm{I}}),\\
&\frac{2^{1/2}}{\pi}\int \mathrm{d}z_{\mathrm{I}} P(z) = \Pi_x(2^{1/2} z_{\mathrm{R}}),\\
&2\int_C \mathrm{d}\mu_{\mathrm{E}}(z) P(z) = \mathbf{1}.
\end{aligned} \tag{48}$$

They are used below to prove analogous marginal properties for the Wigner functions.

We next introduce the displaced number states

$$|N; z\rangle_{\mathrm{dn}} = D(z)|N\rangle_n \tag{49}$$

Here the index 'dn' indicates 'displaced number states'. We call $\mathcal{H}_{\mathrm{even}}(z)$ the Hilbert space spanned by the even displaced number states $|2N; z\rangle_{\mathrm{dn}}$, and $\mathcal{H}_{\mathrm{odd}}(z)$ the Hilbert space spanned by the odd displaced number states $|2N+1; z\rangle_{\mathrm{dn}}$. They are generalizations of the Hilbert spaces $\mathcal{H}_{\mathrm{even}}$ and $\mathcal{H}_{\mathrm{odd}}$ introduced earlier. We call $\Pi_{\mathrm{even}}(z)$ and $\Pi_{\mathrm{odd}}(z)$ the projection operators to the spaces $\mathcal{H}_{\mathrm{even}}(z)$ and $\mathcal{H}_{\mathrm{odd}}(z)$, respectively. It is easily seen that

$$\Pi_{\mathrm{even}}(z) = D(z)\Pi_{\mathrm{even}}[D(z)]^{\dagger}, \qquad \Pi_{\mathrm{odd}}(z) = D(z)\Pi_{\mathrm{odd}}[D(z)]^{\dagger}, \tag{50}$$

and that the displaced parity operators can be written as

$$P(z) = \Pi_{\mathrm{even}}(z) - \Pi_{\mathrm{odd}}(z). \tag{51}$$

3.6. Wigner functions

The displaced parity operators are intimately related to the Wigner functions, as explained in [23]. We define the Wigner function [24, 25] of an operator $\mathcal{U}$ as

$$\begin{aligned}
W_{\mathrm{E}}(z; \mathcal{U}) &= \mathrm{Tr}[\mathcal{U}P(z)] = \mathrm{Tr}[\mathcal{U}\Pi_{\mathrm{even}}(z)] - \mathrm{Tr}[\mathcal{U}\Pi_{\mathrm{odd}}(z)]\\
&= \int \mathrm{d}q\, \mathcal{U}_x(2^{1/2} z_{\mathrm{R}} + q, 2^{1/2} z_{\mathrm{R}} - q)\exp(-\mathrm{i}2^{3/2} z_{\mathrm{I}} q)\\
&= \int \mathrm{d}q\, \mathcal{U}_p(2^{1/2} z_{\mathrm{I}} + q, 2^{1/2} z_{\mathrm{I}} - q)\exp(\mathrm{i}2^{3/2} z_{\mathrm{R}} q).
\end{aligned} \tag{52}$$

If $\mathcal{U}$ is a Hermitian operator (e.g., a density matrix) then the Wigner function is real.

Equation (47) immediately leads to the result that the Wigner and Weyl functions are related through a two-dimensional Fourier transform:

$$W_{\mathrm{E}}(\zeta; \mathcal{U}) = \frac{1}{2}\int \mathrm{d}\mu_{\mathrm{E}}(z)\tilde{W}_{\mathrm{E}}(z; \mathcal{U}) \exp[2\mathrm{i}(\zeta_{\mathrm{I}} z_{\mathrm{R}} - \zeta_{\mathrm{R}} z_{\mathrm{I}})]. \tag{53}$$

This can be used as the starting point for the so-called 'extended phase-space' formalism which has been studied in [105, 106] and is not reviewed here. The operator $\mathcal{U}$ can be expanded in terms of the displaced parity operators with the Wigner functions as coefficients:

$$\mathcal{U} = 4\int_C \mathrm{d}\mu_{\mathrm{E}}(z) W_{\mathrm{E}}(z; \mathcal{U}) P(z). \tag{54}$$

The marginal properties of the displaced parity operators in equation (48) lead straightforwardly to the following marginal properties for the Wigner functions:

$$\begin{aligned}
&\frac{2^{1/2}}{\pi}\int \mathrm{d}z_{\mathrm{R}}\, W_{\mathrm{E}}(z;\mathcal{U}) = \mathcal{U}_p(2^{1/2}z_{\mathrm{I}}, 2^{1/2}z_{\mathrm{I}}),\\
&\frac{2^{1/2}}{\pi}\int \mathrm{d}z_{\mathrm{I}}\, W_{\mathrm{E}}(z;\mathcal{U}) = \mathcal{U}_x(2^{1/2}z_{\mathrm{R}}, 2^{1/2}z_{\mathrm{R}}),\\
&2\int_C \mathrm{d}\mu_{\mathrm{E}}(z) W_{\mathrm{E}}(z;\mathcal{U}) = \mathrm{Tr}[\mathcal{U}].
\end{aligned} \tag{55}$$

If $\mathcal{U}$ is a density matrix, then the right-hand sides of these relations are measurable probability distributions. In this special case, the above relations lead to the interpretation of the Wigner function as a pseudo-probability density. Its integral along the real axis is the probability distribution $\mathcal{U}_p(2^{1/2}z_{\mathrm{I}}, 2^{1/2}z_{\mathrm{I}})$, and its integral along the imaginary axis is the probability distribution $\mathcal{U}_x(2^{1/2}z_{\mathrm{R}}, 2^{1/2}z_{\mathrm{R}})$. At the same time, the relation $W_{\mathrm{E}}(z;\mathcal{U}) = \mathrm{Tr}[\mathcal{U}\Pi_{\mathrm{even}}(z)] - \mathrm{Tr}[\mathcal{U}\Pi_{\mathrm{odd}}(z)]$ shows that the Wigner function is the difference of two real probabilities (and this explains the fact that it takes negative values).

There are more marginal properties for the Wigner functions, which involve $|W_{\mathrm{E}}(z;\mathcal{U})|^2$ [105]:

$$\begin{aligned}
&\frac{2^{1/2}}{\pi}\int \mathrm{d}z_{\mathrm{R}} |W_{\mathrm{E}}(z;\mathcal{U})|^2 = \int \mathrm{d}q |\mathcal{U}_p(2^{1/2}z_{\mathrm{I}}+q, 2^{1/2}z_{\mathrm{I}}-q)|^2,\\
&\frac{2^{1/2}}{\pi}\int \mathrm{d}z_{\mathrm{I}} |W_{\mathrm{E}}(z;\mathcal{U})|^2 = \int \mathrm{d}q |\mathcal{U}_x(2^{1/2}z_{\mathrm{R}}+q, 2^{1/2}z_{\mathrm{R}}-q)|^2,\\
&4\int_C \mathrm{d}\mu_{\mathrm{E}}(z) |W_{\mathrm{E}}(z;\mathcal{U})|^2 = \mathrm{Tr}[\mathcal{U}^\dagger\mathcal{U}].
\end{aligned} \tag{56}$$

3.7. Radon transform

In equations (37), we gave the marginal properties for the displacement operator using integrations along the real and imaginary axes. Of course, there is nothing special about these two axes, and we now give a general relation that involves integration along an arbitrary line $z_{\mathrm{R}}\cos\theta + z_{\mathrm{I}}\sin\theta + 2^{1/2}q = 0$. These relations are the basis for recent work on quantum tomography [107–111] that reconstructs the quantum state of light from a series of measurements.

Multiplication of both sides of equation (37) with $V(\theta)$ and $V(\theta)^\dagger$ gives

$$\begin{aligned}
P_0\Pi_\theta(q) &= \frac{1}{2^{1/2}\pi}\int \mathrm{d}s\, D[(-2^{1/2}q + \mathrm{i}s)\,\mathrm{e}^{\mathrm{i}\theta}]\\
&= 2^{-1/2}\int_C \mathrm{d}\mu_{\mathrm{E}}(z) D(z)\delta(z_{\mathrm{R}}\cos\theta + z_{\mathrm{I}}\sin\theta + 2^{1/2}q),
\end{aligned} \tag{57}$$

where q and s are real numbers. Equation (57) shows that the Radon transform [112] of the displacement operator is the operator $P_0\Pi_\theta(q)$. A Fourier transform in conjunction with equation (47) gives the inverse Radon transform:

$$\int \mathrm{d}q\, \exp(-\mathrm{i}\xi q)\Pi_\theta(q) = \frac{1}{2}D(-2^{-1/2}\mathrm{i}\xi\,\mathrm{e}^{\mathrm{i}\theta}). \tag{58}$$

Taking the trace of both sides with an operator $\mathcal{U}$, we get

$$\int \mathrm{d}q\, \exp(-\mathrm{i}\xi q)\,\mathrm{Tr}[\mathcal{U}\Pi_\theta(q)] = \frac{1}{2}\tilde{W}_{\mathrm{E}}(-2^{-1/2}\mathrm{i}\xi\,\mathrm{e}^{\mathrm{i}\theta};\mathcal{U}). \tag{59}$$

If $\mathcal{U}$ is a density matrix, then the quantity $\mathrm{Tr}[\mathcal{U}\Pi_\theta(q)]$ is measurable with homodyne detection techniques. In this case, the above relation gives the Weyl function of the system from which the Wigner function and the density matrix can be calculated.

3.8. Moyal star product

The Moyal formalism [113, 114] maps quantum-mechanical operators to functions in phase space and provides an autonomous route to quantization. The non-commutativity is introduced by using as a multiplication rule for these functions the Moyal star product. In the limit $\hbar \to 0$, the Moyal star product is equal to the ordinary product plus quantum corrections which are powers of $\hbar$. In this limit, the quantum-mechanical commutator becomes the Poisson bracket of classical mechanics plus quantum corrections which are higher powers of $\hbar$. Therefore, the formalism is suitable for the study of the semiclassical limit of quantum theories [115, 116].

Related to the Moyal formalism is the Stratonovich–Weyl approach [117] which postulates the desirable properties of the map between quantum-mechanical operators and functions in phase space. These approaches have been generalized in the theory of deformations [118], which is not discussed here. Similar philosophy to the Moyal formalism is also used in the Berezin formalism which is discussed later.

The Moyal star product gives the Wigner function of the product of two operators $\mathcal{U}_1$ and $\mathcal{U}_2$ in terms of two Wigner functions of these operators. Using equation (52), we show that

$$W_{\mathrm{E}}(z;\mathcal{U}_1)\star W_{\mathrm{E}}(z;\mathcal{U}_2) \equiv W_{\mathrm{E}}(z;\mathcal{U}_1\mathcal{U}_2) = 4\int_C \mathrm{d}\mu_{\mathrm{E}}(z_1) \times \int_C \mathrm{d}\mu_{\mathrm{E}}(z_2) W_{\mathrm{E}}(z+z_1;\mathcal{U}_1) W_{\mathrm{E}}(z+z_2;\mathcal{U}_2)\exp[2\mathrm{i}(z_1^* z_2 - z_1 z_2^*)]. \tag{60}$$

The properties of the Moyal star product reflect the properties of the operators, i.e., associativity holds, but commutativity does not hold in general. A Taylor expansion leads to the formula

$$W_{\mathrm{E}}(z;\mathcal{U}_1)\star W_{\mathrm{E}}(z;\mathcal{U}_2) = \sum_{M,N} \frac{(-1)^N}{M!N!}\left(\frac{\mathrm{i}}{4}\right)^{M+N} \frac{\partial^{M+N} W_{\mathrm{E}}(z;\mathcal{U}_1)}{\partial z_{\mathrm{R}}^M \partial z_{\mathrm{I}}^N} \frac{\partial^{M+N} W_{\mathrm{E}}(z;\mathcal{U}_2)}{\partial z_{\mathrm{R}}^N \partial z_{\mathrm{I}}^M}. \tag{61}$$

We note that Planck's constant has been taken equal to 1, and does not appear explicitly in our formulae. However, if we want to study the semiclassical limit, we need to attach $\hbar^{1/2}$ to each derivative. In the limit $\hbar \to 0$, only the first term survives and the star product becomes ordinary product

$$W_{\mathrm{E}}(z;\mathcal{U}_1)\star W_{\mathrm{E}}(z;\mathcal{U}_2) \approx W_{\mathrm{E}}(z;\mathcal{U}_1) W_{\mathrm{E}}(z;\mathcal{U}_2). \tag{62}$$

It is seen that when all quantum corrections are turned off, we get the classical result that the operators commute.

We next keep terms of the order $O(\hbar)$ in the expansion of equation (61) and show that that the commutator $[\mathcal{U}_1, \mathcal{U}_2]$ is represented by the Wigner function

$$W_{\mathrm{E}}(z;[\mathcal{U}_1,\mathcal{U}_2]) \approx \mathrm{i}\{W_{\mathrm{E}}(z;\mathcal{U}_1), W_{\mathrm{E}}(z;\mathcal{U}_2)\}_{\mathrm{E}}, \tag{63}$$

where the Poisson bracket has been given earlier in equation (3). It is seen that when the higher order quantum corrections are turned off, the Wigner function of the quantum-mechanical commutator becomes the Poisson bracket of the Wigner functions of the two operators. This demonstrates the correspondence principle in the semiclassical limit.

It should be stressed that the above semiclassical expressions are valid for Wigner functions which are smooth functions of $\hbar$ so that the Taylor expansion used to derive equation (61) is valid. For example, Wigner functions related to chaotic systems can be singular functions of $\hbar$. Such cases require special study and provide a deeper understanding of the relationship between classical and quantum chaos. The semiclassical behaviour of Wigner functions has been discussed in [24, 40].

3.9. Coherent states

Coherent states have been introduced by various authors [119–122] (and also [28]) as

$$|z\rangle_{\rm c} = D(z)|0\rangle = \exp\left(-\frac{1}{2}|z|^2\right)\sum_{N=0}^{\infty} z^N (N!)^{-1/2}|N\rangle_n, \tag{64}$$

where the index 'c' indicates coherent states. They are called Glauber coherent states in order to distinguish them from other coherent states which are discussed later. They are eigenstates of the annihilation operator:

$$a|z\rangle_{\rm c} = z|z\rangle_{\rm c}. \tag{65}$$

Their wavefunction in the x representation is the Gaussian function:

$${}_x\langle x|z\rangle_{\rm c} = \pi^{-1/4}\exp\left[-\tfrac{1}{2}x^2 + 2^{1/2}zx - zz_{\rm R}\right]. \tag{66}$$

Using this we can calculate the expectation values $\langle x\rangle$, $\langle p\rangle$ and the uncertainties:

$$\Delta x = [\langle x^2\rangle - \langle x\rangle^2]^{1/2}, \qquad \Delta p = [\langle p^2\rangle - \langle p\rangle^2]^{1/2}, \qquad \sigma_{xp} = \left\langle \tfrac{1}{2}(xp+px)\right\rangle - \langle x\rangle\langle p\rangle. \tag{67}$$

The result is

$$\langle x\rangle = 2^{1/2}z_{\rm R}, \qquad \langle p\rangle = 2^{1/2}z_{\rm I}, \qquad \Delta x = \Delta p = 2^{-1/2}, \qquad \sigma_{xp} = 0. \tag{68}$$

The coherent states are minimum uncertainty states in the sense that $\Delta x\Delta p = 1/2$, and in addition to that $\Delta x/\Delta p = 1$. The uncertainty ellipse of the coherent state $|z\rangle$ in the x–p phase space is a circle with centre at $(2^{1/2}z_{\rm R}, 2^{1/2}z_{\rm I})$ and radius $2^{-1/2}$.

The overlap of two coherent states is given by

$${}_{\rm c}\langle z|w\rangle_{\rm c} = \exp\left(-\tfrac{1}{2}|z|^2 - \tfrac{1}{2}|w|^2 + z^*w\right). \tag{69}$$

$|{}_{\rm c}\langle z|w\rangle_{\rm c}|^2$ is a function of the Euclidean distance $|z-w|$:

$$\mathcal{G}_{\rm E}(z,w) \equiv |{}_{\rm c}\langle z|w\rangle_{\rm c}|^2 = \exp(-|z-w|^2). \tag{70}$$

An important property of coherent states is the resolution of the identity:

$$\int_C {\rm d}\mu_{\rm E}(z)|z\rangle_{\rm c}\,{}_{\rm c}\langle z| = \mathbf{1}. \tag{71}$$

Using this we can expand an arbitrary state $|f\rangle$ in terms of coherent states as

$$|f\rangle = \int_C {\rm d}\mu_{\rm E}(z) f(z)|z\rangle_{\rm c}, \qquad f(z) = {}_{\rm c}\langle z|f\rangle. \tag{72}$$

This shows that the set of coherent states is at least complete; in fact, it is 'highly overcomplete' in the sense that it will be shown later that some very small subsets of coherent states are also overcomplete.

Another property of coherent states is the 'temporal stability' [15]. For a class of Hamiltonians, coherent states evolve into other coherent states. This is intimately related with the fact that the operators used in the construction of coherent states form a group. For the Glauber coherent states that we study here, the displacement operators form the Heisenberg–Weyl group, and the relevant Hamiltonian is the 'displaced harmonic oscillator' Hamiltonian:

$$H = D(z)[\omega a^\dagger a][D(z)]^\dagger = \omega[a^\dagger a - za^\dagger - z^*a + |z^2|]. \tag{73}$$

Indeed, it is easily seen that

$$\exp(\mathrm{i}Ht)|\zeta\rangle_{\mathrm{c}} = \exp[\mathrm{i}|z|^2 \sin\omega t]|\zeta\, \mathrm{e}^{\mathrm{i}\omega t} + z(1-\mathrm{e}^{\mathrm{i}\omega t})\rangle_{\mathrm{c}}. \tag{74}$$

A direct consequence of the marginal properties of the displacements operators given in equation (37) is the following relations:

$$\begin{aligned}
&\int \mathrm{d}z_{\mathrm{R}}|z\rangle_{\mathrm{c}} = 2^{1/2}\pi^{3/4}\exp\left(-\frac{1}{4}z_{\mathrm{I}}^2\right)|2^{-1/2}z_{\mathrm{I}}\rangle_p,\\
&\int \mathrm{d}z_{\mathrm{I}}|z\rangle_{\mathrm{c}} = 2^{1/2}\pi^{3/4}\exp\left(-\frac{1}{4}z_{\mathrm{R}}^2\right)|2^{-1/2}z_{\mathrm{R}}\rangle_x,\\
&\int \mathrm{d}\mu_{\mathrm{E}}(z)|z\rangle_{\mathrm{c}} = 2|0\rangle_n.
\end{aligned} \tag{75}$$

More generally, integration along an arbitrary line can be evaluated using equation (57):

$$\int \mathrm{d}\mu_{\mathrm{E}}(z)\delta(z_{\mathrm{R}}\cos\theta + z_{\mathrm{I}}\sin\theta + 2^{1/2}q)|z\rangle_{\mathrm{c}} = 2^{1/2}\pi^{-1/4}\exp\left(-\frac{1}{2}q^2\right)|-q\rangle_\theta. \tag{76}$$

They are used below to derive marginal properties for the Bargmann functions.

3.10. P and Q functions

The Q and P representations [123–125] of a bounded operator $\mathcal{U}$ are defined as

$$Q_{\mathrm{E}}(z;\mathcal{U}) \equiv {}_{\mathrm{c}}\langle z|\mathcal{U}|z\rangle_{\mathrm{c}}, \tag{77}$$

$$\mathcal{U} = \int_C \mathrm{d}\mu_{\mathrm{E}}(z)P_{\mathrm{E}}(z;\mathcal{U})|z\rangle_{\mathrm{c}\,\mathrm{c}}\langle z|. \tag{78}$$

Berezin used the terms covariant and contravariant symbols, respectively, for these quantities.

The P function can be calculated through the following two-dimensional Fourier transform derived by Mehta [126]:

$$P_{\mathrm{E}}(z;\mathcal{U}) = \exp(|z|^2)\int_C \mathrm{d}\mu_{\mathrm{E}}(w)\,{}_{\mathrm{c}}\langle -w|\mathcal{U}|w\rangle_{\mathrm{c}}\exp(|w|^2)\exp[zw^* - z^*w]. \tag{79}$$

This integral involves the two-dimensional Fourier transform of ${}_{\mathrm{c}}\langle -w|\mathcal{U}|w\rangle_{\mathrm{c}}\exp(|w|^2)$. In many cases, it only exists in the context of generalized function theory and the corresponding P function is singular. For example, if ${}_{\mathrm{c}}\langle -w|\mathcal{U}|w\rangle_{\mathrm{c}}\exp(|w|^2)$ is a polynomial function of w, then its Fourier transform involves derivatives of delta functions (tempered distributions); if it is an exponential function of w, then its Fourier transform involves more general Schwartz distributions. The existence of the P function has been discussed in [2, 127].

We can easily show that that the trace of an operator $\mathcal{U}$ is given by

$$\mathrm{Tr}\,\mathcal{U} = \int_C Q_{\mathrm{E}}(z;\mathcal{U})\,\mathrm{d}\mu_{\mathrm{E}}(z) = \int_C P_{\mathrm{E}}(z;\mathcal{U})\,\mathrm{d}\mu_{\mathrm{E}}(z) \tag{80}$$

and also that the trace of the product of two operators is given by

$$\mathrm{Tr}(\mathcal{U}_1\mathcal{U}_2) = \int_C Q_{\mathrm{E}}(z;\mathcal{U}_1)P_{\mathrm{E}}(z;\mathcal{U}_2)\,\mathrm{d}\mu_{\mathrm{E}}(z). \tag{81}$$

There is a well-known connection [128] between the various orderings of an operator $\Theta(a, a^\dagger)$ and the Q, P and Wigner representations of this operator. We express $\Theta(a, a^\dagger)$ as

$$\Theta(a, a^\dagger) = \sum_{M,N} f_{NM}(a^\dagger)^N a^M = \sum_{M,N} g_{NM}a^M(a^\dagger)^N = \sum_{M,N} s_{NM}\{a^M(a^\dagger)^N\}_W. \tag{82}$$

Here $\{\}_W$ indicates the Weyl-symmetric product which can be written as

$$\{a^M(a^\dagger)^N\}_W = \left[\partial_\beta^M \partial_\gamma^N \, \mathrm{e}^{\beta a+\gamma a^\dagger}\right]_{\beta=\gamma=0}. \tag{83}$$

We can show that

$$Q_{\mathrm{E}}(z;\Theta) = \sum_{M,N} f_{NM}(z^*)^N z^M, \qquad P_{\mathrm{E}}(z;\Theta) = \sum_{M,N} g_{NM}(z^*)^N z^M,$$
$$W_{\mathrm{E}}(z;\Theta) = \sum_{M,N} s_{NM}(z^*)^N z^M. \tag{84}$$

It is seen that the P, Q and Wigner functions are different from each other, because of the non-commutativity of the operators a and $a^\dagger$. In the semiclassical limit $\hbar \to 0$, the non-commutativity is turned off, the coefficients f_{NM}, g_{NM} and s_{NM} become equal to each other, and consequently the P, Q and Wigner functions become equal to each other. This is discussed in a more formal way below.

We next present some interesting relations between the Wigner function and the P and Q representations of an operator. We first show that for $\lambda > 0$

$$\lambda \int_C \mathrm{d}\mu_{\mathrm{E}}(w)\mathcal{G}_{\mathrm{E}}(\lambda^{1/2}z, \lambda^{1/2}w) f(w, w^*) = \exp\left[\frac{1}{4\lambda}\Delta_z^{(\mathrm{E})}\right] f(z, z^*), \tag{85}$$

where the Laplace operator has been given in equation (5) and $\mathcal{G}_{\mathrm{E}}(z, w)$ has been given in equation (70). This can easily be proved by taking the Fourier transform of both sides of this equation. We then show the following relations:

$$Q_{\mathrm{E}}(z;\mathcal{U}) = \int_C \mathrm{d}\mu_{\mathrm{E}}(z)\,\mathcal{G}_{\mathrm{E}}(z,w)P_{\mathrm{E}}(z;\mathcal{U}) = \exp\left[\frac{1}{4}\Delta_z^{(\mathrm{E})}\right]P_{\mathrm{E}}(z;\mathcal{U}),$$
$$Q_{\mathrm{E}}(z;\mathcal{U}) = 2\int_C \mathrm{d}\mu_{\mathrm{E}}(z)\,\mathcal{G}_{\mathrm{E}}(2^{1/2}z, 2^{1/2}w)W_{\mathrm{E}}(z;\mathcal{U}) = \exp\left[\frac{1}{8}\Delta_z^{(\mathrm{E})}\right]W_{\mathrm{E}}(z;\mathcal{U}), \tag{86}$$
$$W_{\mathrm{E}}(z;\mathcal{U}) = 2\int_C \mathrm{d}\mu_{\mathrm{E}}(z)\,\mathcal{G}_{\mathrm{E}}(2^{1/2}z, 2^{1/2}w)P_{\mathrm{E}}(z;\mathcal{U}) = \exp\left[\frac{1}{8}\Delta_z^{(\mathrm{E})}\right]P_{\mathrm{E}}(z;\mathcal{U}).$$

We have explained earlier that for semiclassical studies we need to attach $\hbar^{1/2}$ to each derivative. A Taylor expansion of the exponentials shows that in the limit $\hbar \to 0$, only the first term survives and the P, Q and Wigner functions become equal to each other. If we keep the first two terms in the Taylor expansion, we get

$$W_{\mathrm{E}}(z;\mathcal{U}) \approx \left[1+\tfrac{1}{8}\Delta_z^{(\mathrm{E})}\right]P_{\mathrm{E}}(z;\mathcal{U}),$$
$$Q_{\mathrm{E}}(z;\mathcal{U}) \approx \left[1+\tfrac{1}{8}\Delta_z^{(\mathrm{E})}\right]W_{\mathrm{E}}(z;\mathcal{U}) \approx \left[1+\tfrac{1}{4}\Delta_z^{(\mathrm{E})}\right]P_{\mathrm{E}}(z;\mathcal{U}). \tag{87}$$

These semiclassical expressions assume that the P, Q and Wigner functions are smooth functions of $\hbar$ so that the Taylor expansions are valid. We have already made this comment earlier in conjunction with the semiclassical expressions of equations (62) and (63), and the same comment applies here also.

4. Bargmann analytic representation in the complex plane

4.1. States

We consider an arbitrary (normalized) state

$$|f\rangle = \sum_{N=0}^{\infty} f_N |N\rangle_n, \qquad \sum_{N=0}^{\infty} |f_N|^2 = 1 \tag{88}$$

and use the notation

$$\langle f| = \sum_{N=0}^{\infty} f_N^*\,{}_n\langle N|, \qquad |f^*\rangle = \sum_{N=0}^{\infty} f_N^* |N\rangle_n, \qquad \langle f^*| = \sum_{N=0}^{\infty} f_N\,{}_n\langle N|. \tag{89}$$

In the Bargmann representation [28, 29], the state $|f\rangle$ is represented by the function

$$f_{\mathrm{E}}(z) = \exp\left[\frac{1}{2}|z|^2\right]\,{}_{\mathrm{c}}\langle z^*|f\rangle = \exp\left(\frac{1}{2}|z|^2\right)\langle f^*|z\rangle_{\mathrm{c}} = \sum_{N=0}^{\infty} f_N z^N (N!)^{-1/2}, \tag{90}$$

which is analytic in the complex plane C (entire function). The index 'E' stands for Euclidean.

Let ℓ_0 be an anticlockwise contour around the origin and ℓ be any anticlockwise contour. The fact that $f_{\mathrm{E}}(z)$ is analytic leads to the following relation that gives the coefficients f_N in terms of contour integrals:

$$\oint_{\ell_0} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{f_{\mathrm{E}}(z)(N!)^{1/2}}{z^{N+1}} = f_N. \tag{91}$$

Let w_i be the zeros of $f_{\mathrm{E}}(z)$ (i.e., $f_{\mathrm{E}}(w_i) = 0$) in the interior of the contour ℓ and p_i be the multiplicities of these zeros. It can be shown that

$$\oint_{\ell} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\partial_z f_{\mathrm{E}}(z)}{f_{\mathrm{E}}(z)} z^N = \sum_i p_i w_i^N. \tag{92}$$

A special case of this result is that

$$\oint_{\ell} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\partial_z f_{\mathrm{E}}(z)}{f_{\mathrm{E}}(z)} = \mathcal{N}_{\mathrm{zeros}}(\ell), \tag{93}$$

where $\mathcal{N}_{\mathrm{zeros}}(\ell)$ is the number of zeros of the function $f_{\mathrm{E}}(z)$ inside the contour ℓ (taking into account the multiplicities). The physical significance of these zeros is discussed later.

Using the resolution of the identity in equation (71), we show that the scalar product is given by

$$\langle f|g\rangle = \int_C [f_{\mathrm{E}}(z)]^* g_{\mathrm{E}}(z) \exp(-|z|^2)\,\mathrm{d}\mu_{\mathrm{E}}(z) = \sum_N f_N^* g_N. \tag{94}$$

As an example, we consider the coherent state $|w\rangle_{\mathrm{c}}$ for which the Bargmann function is

$$f_{\mathrm{E}}(z) = \exp\left[wz - \tfrac{1}{2}|w|^2\right]. \tag{95}$$

Another examples are the squeezed states which are defined as

$$|w; r, \theta\rangle_{\mathrm{sq}} = S(r,\theta)|w\rangle_{\mathrm{c}}, \qquad S(r,\theta) = \exp\left[-\tfrac{1}{4} r\,\mathrm{e}^{-\mathrm{i}\theta}(a^\dagger)^2 + \tfrac{1}{4} r\,\mathrm{e}^{\mathrm{i}\theta} a^2\right], \tag{96}$$

where the index 'sq' indicates squeezed states. In this case, the Bargmann function is

$$f_{\mathrm{E}}(z) = (1 - |\alpha|^2)^{1/4} \exp\left[\tfrac{1}{2}\alpha z^2 + \beta z + \gamma\right], \tag{97}$$

where

$$\alpha = -\tanh\left(\frac{r}{2}\right)\mathrm{e}^{-\mathrm{i}\theta}, \qquad \beta = w(1 - |\alpha|^2)^{1/2}, \qquad \gamma = -\frac{1}{2}\alpha^* w^2 - \frac{1}{2}|w|^2. \tag{98}$$

The creation and annihilation operators are represented as

$$a \to \partial_z, \qquad a^\dagger \to z. \tag{99}$$

Therefore, the action of the general displacement operator $D(w)$ of equation (32) on a state $|f\rangle$ represented with the function $f_{\mathrm{E}}(z)$ is given by

$$D(w)|f\rangle \to \exp\left(-\tfrac{1}{2}|z|^2 + wz\right) f_{\mathrm{E}}(z - w^*). \tag{100}$$

4.2. Marginal properties and physical interpretation of the Bargmann functions

The marginal properties for coherent states given in equation (75) lead to the following marginal properties for the Bargmann function:

$$\begin{aligned}\int \mathrm{d}z_{\mathrm{R}}\, f_{\mathrm{E}}(z) \exp\left(-\frac{1}{2}z_{\mathrm{R}}^2\right) &= 2^{1/2}\pi^{3/4} \exp\left(\frac{1}{4}z_{\mathrm{I}}^2\right) f_p(-2^{-1/2}z_{\mathrm{I}}),\\ \int \mathrm{d}z_{\mathrm{I}}\, f_{\mathrm{E}}(z) \exp\left(-\frac{1}{2}z_{\mathrm{I}}^2\right) &= 2^{1/2}\pi^{3/4} \exp\left(\frac{1}{4}z_{\mathrm{R}}^2\right) f_x(2^{-1/2}z_{\mathrm{R}}).\end{aligned} \tag{101}$$

The Bargmann function can be interpreted as a (complex) density of the wavefunction in phase space in the sense that its integral (with Gaussian weight) along the real axis is the wavefunction f_p and its integral along the imaginary axis is the wavefunction f_x. We note however that f_x and f_p are not independent; they are related through a Fourier transform. This is reminiscent of the interpretation of the Wigner function as a pseudo-probability density in the sense that its integral along the real axis is the probability distribution $|f_p|^2$ and its integral along the imaginary axis is the probability distribution $|f_x|^2$.

A more general relation can be derived using equation (76) which involves integration along an arbitrary line

$$\begin{aligned}&\int \mathrm{d}\mu_{\mathrm{E}}(z)\, \delta(z_{\mathrm{R}}\cos\theta + z_{\mathrm{I}}\sin\theta + 2^{1/2}q)\, f_{\mathrm{E}}(z) \exp\left(-\frac{1}{2}|z|^2\right)\\ &\qquad = 2^{1/2}\pi^{-1/4} \exp\left(-\frac{1}{2}q^2\right) f_{-\theta}(-q).\end{aligned} \tag{102}$$

4.3. Operators

An operator $\mathcal{U}$ with matrix elements $\mathcal{U}_{MN} = {}_n\langle M|\mathcal{U}|N\rangle_n$ is represented by the following Bargmann kernel:

$$\mathcal{K}_{\mathrm{E}}(z, \zeta^*; \mathcal{U}) = \exp\left[\frac{1}{2}(|z|^2 + |\zeta|^2)\right] {}_{\mathrm{c}}\langle z^*|\mathcal{U}|\zeta^*\rangle_{\mathrm{c}} = \sum_{M,N} \mathcal{U}_{MN} z^M (\zeta^*)^N. \tag{103}$$

The index 'E' in the notation stands for Euclidean. The Bargmann function corresponding to the state $|s\rangle = \mathcal{U}|f\rangle$ is

$$s_{\mathrm{E}}(z) = \int_C \mathrm{d}\mu_{\mathrm{E}}(\zeta)\, \mathrm{e}^{-|\zeta|^2} \mathcal{K}_{\mathrm{E}}(z, \zeta^*; \mathcal{U})\, f_{\mathrm{E}}(\zeta). \tag{104}$$

The kernel $\mathcal{K}_{\mathrm{E}}(z, \zeta^*; \mathcal{U})$ is an analytic function in the complex plane in both variables z and ζ^*. Consequently, its diagonal component $\mathcal{K}_{\mathrm{E}}(z, z^*; \mathcal{U})$ determines uniquely through analytic continuation $\mathcal{K}_{\mathrm{E}}(z, \zeta^*; \mathcal{U})$.

As examples, we consider the operators $\mathbf{1}$, a, $a^\dagger$ and the displacement operator $D(w)$ and we find

$$\begin{aligned}&\mathcal{K}_{\mathrm{E}}(z, \zeta^*; \mathbf{1}) = \exp(z\zeta^*),\\ &\mathcal{K}_{\mathrm{E}}(z, \zeta^*; a) = \zeta^* \exp(z\zeta^*),\\ &\mathcal{K}_{\mathrm{E}}(z, \zeta^*; a^\dagger) = z \exp(z\zeta^*),\\ &\mathcal{K}_{\mathrm{E}}(z, \zeta^*; D(w)) = \exp\left(-\tfrac{1}{2}|w|^2 + wz - w^*\zeta^* + z\zeta^*\right).\end{aligned} \tag{105}$$

$\mathcal{K}_{\mathrm{E}}(z, \zeta^*; \mathbf{1})$ is called the reproducing kernel, because when it acts on a state it gives the same state:

$$\int_C \mathrm{d}\mu_{\mathrm{E}}(\zeta)\, \mathrm{e}^{-|\zeta|^2 + z\zeta^*} f_{\mathrm{E}}(\zeta) = f_{\mathrm{E}}(z). \tag{106}$$

For the creation and annihilation operators, we have given a differential representation in equation (99) and an integral representation in equation (105). We can check that they are consistent by proving the relations:

$$\begin{aligned}
&\int_C \mathrm{d}\mu_{\mathrm{E}}(\zeta)\,\mathrm{e}^{-|\zeta|^2}\mathcal{K}_{\mathrm{E}}(z,\zeta^*;a)f_{\mathrm{E}}(\zeta) = \partial_z f_{\mathrm{E}}(z),\\
&\int_C \mathrm{d}\mu_{\mathrm{E}}(\zeta)\,\mathrm{e}^{-|\zeta|^2}\mathcal{K}_{\mathrm{E}}(z,\zeta^*;a^\dagger)f_{\mathrm{E}}(\zeta) = zf_{\mathrm{E}}(z).
\end{aligned} \tag{107}$$

The trace of an operator is given by the formula

$$\mathrm{Tr}\,\mathcal{U} = \int_C \mathrm{d}\mu_{\mathrm{E}}(z)\,\mathrm{e}^{-|z|^2}\mathcal{K}_{\mathrm{E}}(z,z^*;\mathcal{U}). \tag{108}$$

The product $\mathcal{U}_1\mathcal{U}_2$ of two operators is represented by the kernel

$$\mathcal{K}_{\mathrm{E}}(z,\zeta^*;\mathcal{U}_1\mathcal{U}_2) = \int_C \mathrm{d}\mu_{\mathrm{E}}(w)\,\mathrm{e}^{-|w|^2}\mathcal{K}_{\mathrm{E}}(z,w^*;\mathcal{U}_1)\mathcal{K}_{\mathrm{E}}(w,\zeta^*;\mathcal{U}_2). \tag{109}$$

There are various relations that connect the Bargmann kernel of an operator with its P, Q, Wigner and Weyl functions. We first prove that

$$\begin{aligned}
\mathcal{K}_{\mathrm{E}}(z,\zeta^*;\mathcal{U}) &= \int_C \mathrm{d}\mu_{\mathrm{E}}(w)P_{\mathrm{E}}(w;\mathcal{U})\exp(wz + w^*\zeta^* - |w|^2)\\
&= \mathrm{e}^{z\zeta^*}\int_C \mathrm{d}\mu_{\mathrm{E}}(w)\tilde{W}_{\mathrm{E}}(w;\mathcal{U})\exp\left[-wz + w^*\zeta^* - \frac{1}{2}|w|^2\right].
\end{aligned} \tag{110}$$

Using equation (86) we easily show the following relations that involve the 'diagonal part' of the Bargmann kernel:

$$\begin{aligned}
\mathcal{K}_{\mathrm{E}}(z^*,z;\mathcal{U}) &= \mathrm{e}^{|z|^2}Q_{\mathrm{E}}(z;\mathcal{U}) = \mathrm{e}^{|z|^2}\exp\left[\tfrac{1}{4}\Delta_z^{(\mathrm{E})}\right]P_{\mathrm{E}}(z;\mathcal{U})\\
&= \mathrm{e}^{|z|^2}\exp\left[\tfrac{1}{8}\Delta_z^{(\mathrm{E})}\right]W_{\mathrm{E}}(z;\mathcal{U}).
\end{aligned} \tag{111}$$

In the semiclassical limit, we use a Taylor expansion (as in equation (87)) and we get

$$\begin{aligned}
\mathcal{K}_{\mathrm{E}}(z^*,z;\mathcal{U}) &= \mathrm{e}^{|z|^2}Q_{\mathrm{E}}(z;\mathcal{U}) \approx \mathrm{e}^{|z|^2}\left[1 + \tfrac{1}{4}\Delta_z^{(\mathrm{E})}\right]P_{\mathrm{E}}(z;\mathcal{U})\\
&\approx \mathrm{e}^{|z|^2}\left[1 + \tfrac{1}{8}\Delta_z^{(\mathrm{E})}\right]W_{\mathrm{E}}(z;\mathcal{U}).
\end{aligned} \tag{112}$$

Equation (79) can be expressed in terms of the Bargmann kernel as

$$P_{\mathrm{E}}(z;\mathcal{U}) = \mathrm{e}^{|z|^2}\int_C \mathrm{d}\mu_{\mathrm{E}}(w)\mathcal{K}_{\mathrm{E}}(-w^*,w;\mathcal{U})\exp[zw^* - z^*w]. \tag{113}$$

We next point out that we can define slightly different kernels for the representation of the various operators within the Bargmann formalism. With a trivial revision of the above formulae, they all lead to the same final results. For example, below we will use the following kernel:

$$\mathcal{L}_{\mathrm{E}}(z,\zeta^*;\mathcal{U}) = \mathcal{K}_{\mathrm{E}}(z,\zeta^*;\mathcal{U})\,\mathrm{e}^{-z\zeta^*}. \tag{114}$$

4.4. Euclidean Berezin formalism

The Berezin formalism [30] represents operators with their Bargmann $\mathcal{L}$-kernels of equation (114). It is based on similar philosophy as the Moyal star product formalism, but it uses analytic functions and the technical details are different. It shows that in the limit $\hbar \to 0$ the $\mathcal{L}$-kernel of the product of two operators is equal to the product of the $\mathcal{L}$-kernels of the two operators plus quantum corrections which are powers of $\hbar$. In this limit, the $\mathcal{L}$-kernel of the quantum-mechanical commutator of two operators becomes the Poisson bracket of classical mechanics plus quantum corrections which are higher powers of $\hbar$.

Berezin studied extensively this approach for homogeneous Kähler manifolds. Here we discuss briefly this formalism for the Euclidean plane, and later we discuss it for the unit disc and the extended complex plane.

Using equations (108) and (114), we show that diagonal part of the kernel representing the product $\mathcal{U}_1\mathcal{U}_2$ of two operators is given by

$$\mathcal{L}_{\mathrm{E}}(z, z^*; \mathcal{U}_1\mathcal{U}_2) = \int_C \mathrm{d}\mu_{\mathrm{E}}(w)\, \mathcal{G}_{\mathrm{E}}(z, w)\mathcal{L}_{\mathrm{E}}(z, w^*; \mathcal{U}_1)\mathcal{L}_{\mathrm{E}}(w, z^*; \mathcal{U}_2). \tag{115}$$

We next use equation (85) to show that

$$\mathcal{L}_{\mathrm{E}}(z, z^*; \mathcal{U}_1\mathcal{U}_2) = \left[\exp\left(\tfrac{1}{4}\Delta_\zeta^{(\mathrm{E})}\right)\mathcal{L}_{\mathrm{E}}(z, \zeta^*; \mathcal{U}_1)\mathcal{L}_{\mathrm{E}}(\zeta, z^*; \mathcal{U}_2)\right]_{\zeta=z}. \tag{116}$$

Expansion of the exponential gives

$$\mathcal{L}_{\mathrm{E}}(z, z^*; \mathcal{U}_1\mathcal{U}_2) = \mathcal{L}_{\mathrm{E}}(z, z^*; \mathcal{U}_1)\mathcal{L}_{\mathrm{E}}(z, z^*; \mathcal{U}_2) + \frac{\partial\mathcal{L}_{\mathrm{E}}(z, z^*; \mathcal{U}_1)}{\partial z^*}\frac{\partial\mathcal{L}_{\mathrm{E}}(z, z^*; \mathcal{U}_2)}{\partial z} + \cdots. \tag{117}$$

We have explained earlier that for semiclassical studies we need to attach $\hbar^{1/2}$ to each derivative. In the limit $\hbar \to 0$, only the first term survives and we get

$$\mathcal{L}_{\mathrm{E}}(z, z^*; \mathcal{U}_1\mathcal{U}_2) \approx \mathcal{L}_{\mathrm{E}}(z, z^*; \mathcal{U}_1)\mathcal{L}_{\mathrm{E}}(z, z^*; \mathcal{U}_2). \tag{118}$$

It is seen that when all quantum corrections are turned off, we get the classical result that the operators commute.

We next keep the first two terms in the expansion of equation (117) and show that that the commutator $[\mathcal{U}_1, \mathcal{U}_2]$ is represented by the function

$$\mathcal{L}_{\mathrm{E}}(z, z^*; [\mathcal{U}_1, \mathcal{U}_2]) \approx -\mathrm{i}\{\mathcal{L}_{\mathrm{E}}(z, z^*; \mathcal{U}_1), \mathcal{L}_{\mathrm{E}}(z, z^*; \mathcal{U}_2)\}_{\mathrm{E}}, \tag{119}$$

where the Poisson bracket has been given earlier in equation (3). It is seen that when the higher order quantum corrections are turned off, the quantum-mechanical commutator becomes the Poisson bracket of classical mechanics. These results (and also the results about the Moyal star product in the semiclassical limit discussed earlier in equations (62) and (63)) demonstrate the correspondence principle.

As we have already mentioned in conjunction with the Wigner function, these semiclassical expressions assume that the Bargmann $\mathcal{L}$-kernel is smooth function of $\hbar$ so that the expansion of equation (117) is valid.

5. Growth and zeros of Bargmann functions

5.1. The growth of Bargmann functions

The growth of an entire function $f(z)$ is described by a pair of non-negative numbers (ρ, σ). The order ρ and the type σ are defined as [31]

$$\rho = \lim_{R\to\infty} \sup \frac{\ln\ln M(R)}{\ln R}, \qquad \sigma = \lim_{R\to\infty} \sup \frac{\ln M(R)}{R^\rho}, \tag{120}$$

where $M(R)$ is the maximum value of $|f(z)|$ on the circle $|z| = R$. In a simple language, a function $f(z)$ with growth (ρ, σ) grows at large distances R from the centre as

$$|f| \approx \exp(\sigma R^\rho). \tag{121}$$

We consider an entire function with growth (ρ_1, σ_1). We say that the growth of this function is larger than (ρ_0, σ_0) if $\rho_1 > \rho_0$ or if $\rho_1 = \rho_0$ and also $\sigma_1 > \sigma_0$. We say that the growth of this function is smaller than (ρ_0, σ_0) if $\rho_1 < \rho_0$ or if $\rho_1 = \rho_0$ and also $\sigma_1 < \sigma_0$. Convergence of the integral in the scalar product formula in equation (94) leads to the conclusion that the growth of Bargmann functions is smaller than $(\rho = 2, \sigma = 1/2)$. The growth of the Bargmann kernel $\mathcal{K}_B(z, \zeta^*; \mathcal{U})$ is also smaller than $(\rho = 2, \sigma = 1/2)$ with respect to both variables z and ζ^*.

Given an entire function

$$f(z) = \sum_{N=0}^{\infty} c_N z^N, \qquad \lim_{N\to\infty} |c_N|^{1/N} = 0, \tag{122}$$

the following formulae are useful in calculations of its growth:

$$\rho = \lim_{N\to\infty} \frac{N \ln N}{-\ln|c_N|}, \qquad \sigma = \frac{1}{e\rho} \lim_{N\to\infty} N|c_N|^{\rho/N}. \tag{123}$$

We consider various examples. The first is an arbitrary superposition of a finite number of number eigenstates. In this case, the Bargmann function is a (finite) polynomial, which is of order $\rho = 0$. For the coherent state $|w\rangle$, the Bargmann function is given in equation (95) and the order is $\rho = 1$ and the type is $\sigma = |w|$. For the squeezed states of equation (97), the order is $\rho = 2$ and the type is $\sigma = \frac{1}{2}\tanh\left(\frac{r}{2}\right)$.

We also give an example of a state which has Bargmann function with a given order ρ and given type σ. It is the state

$$|\rho, \sigma\rangle \equiv \mathcal{N} \sum_{N=0}^{\infty} \frac{e^{i\theta_N} \sigma^{N/\rho} (N!)^{1/2}}{\Gamma\left(\frac{N}{\rho} + 1\right)} |N\rangle_n, \qquad \mathcal{N} = \left[\sum_{N=0}^{\infty} \frac{\sigma^{2N/\rho} N!}{\left[\Gamma\left(\frac{N}{\rho} + 1\right)\right]^2}\right]^{-\frac{1}{2}}, \tag{124}$$

where $\{\theta_N\}$ are arbitrary phases. The normalization constant is finite when $0 \leqslant \rho < 2$ and also when $\rho = 2$ and $\sigma < \frac{1}{2}$.

5.2. Zeros of Bargmann functions

There is a deep connection between the density of zeros of analytic functions and their growth [31], and this has important implications for coherent states and their completeness properties [32–38]. We first discuss the physical significance of the zeros of Bargmann functions.

We consider a state $|f\rangle$ with Bargmann function $f_E(z)$. If ζ is a zero of this function, then equation (90) shows that the coherent state $|\zeta\rangle_c$ is orthogonal to $|f^*\rangle$. A more general result is that if ζ is a zero of $f_E(z)$ with multiplicity M, then $|f^*\rangle$ is orthogonal to the states $\mathcal{N}_N a^{\dagger N} |\zeta\rangle_c$ with $N = 0, \ldots, M-1$. Here $\mathcal{N}_N$ is a normalization coefficient. This can be seen from the relation

$$\left[\partial_z^N f_E(z)\right]_{z=\zeta} = 0 \to \langle f^*|a^{\dagger N}|\zeta\rangle_c = 0. \tag{125}$$

We also express this result in a slightly different way, in terms of the displaced number states of equation (49). If ζ is a zero of $f_E(z)$ with multiplicity M, then $|f^*\rangle$ is orthogonal to the displaced number states $|N;\zeta\rangle_{dn}$ with $N = 0, \ldots, M-1$. This can be seen from the equation

$$\left[\frac{(\partial_z - \zeta)^N}{(N!)^{1/2}} f_B(z)\right]_{z=\zeta} = 0 \to \langle f^*|N;\zeta\rangle_{dn} = 0. \tag{126}$$

We next consider a Bargmann function with an infinite number of zeros. If the sequence of these zeros converges to a (finite) point w in the complex plane, then the Bargmann function is zero everywhere. We will study in detail the case where the zeros $\zeta_1, \zeta_2, \zeta_3, \ldots$ are such that

$$0 < |\zeta_1| \leqslant |\zeta_2| \leqslant |\zeta_3| \leqslant \cdots, \qquad \lim_{N\to\infty} |\zeta_N| = \infty. \tag{127}$$

In order to quantify the density of these zeros, we call $n(R)$ be the number of terms of this sequence enclosed within the circle $|z| < R$. The density of this sequence of complex numbers is described by a pair of non-negative numbers (η, δ) defined as

$$\eta = \lim_{R\to\infty} \sup \frac{\ln n(R)}{\ln R}, \qquad \delta = \lim_{R\to\infty} \frac{n(R)}{R^{\eta}}. \tag{128}$$

In other words, in a sequence of the type of equation (127) with density (η, δ), the number of terms enclosed in a large circle of radius R is

$$n(R) \approx \delta R^{\eta}. \tag{129}$$

It is known [31] that η is the infimum of positive numbers λ for which

$$\sum_{N=1}^{\infty} |\zeta_N|^{-\lambda} < \infty. \tag{130}$$

For this reason, η is called convergence exponent. We note that if we add or subtract a *finite* number of complex numbers to the sequence, it does not change its density.

An example of a sequence of complex numbers with $\eta = \eta_0$ and $\delta = \delta_0$ is

$$\zeta_N = \left(\frac{N}{\delta_0}\right)^{1/\eta_0} \mathrm{e}^{\mathrm{i}\theta_N}, \tag{131}$$

where θ_N are arbitrary phases.

We consider a sequence of the type of equation (127) with density (η_1, δ_1). We say that the density of this sequence is larger than (η_0, δ_0) if $\eta_1 > \eta_0$ or if $\eta_1 = \eta_0$ and also $\delta_1 > \delta_0$. We say that the density of this sequence is smaller than (η_0, δ_0) if $\eta_1 < \eta_0$ or if $\eta_1 = \eta_0$ and also $\delta_1 < \delta_0$.

The connection between the growth of analytic functions and the density of their zeros is described with the relations

$$\eta \leqslant \rho, \qquad \sigma\rho \leqslant \delta. \tag{132}$$

5.3. Hadamard's theorem and its physical meaning

In this section, we construct a Bargmann function with prescribed zeros $\{\zeta_N\}$ of the type given in equation (127). The general answer to this is given by Hadamard's theorem [31] which is a refined version of the Weierstrass theorem and which represents all entire functions as products of Weierstrass factors.

The physical meaning of this is to construct a state $|f\rangle$ which is orthogonal to a given set of coherent states $\{|\zeta_N\rangle\}$. At this stage, we assume that the set of coherent states $\{|\zeta_N\rangle\}$ is undercomplete, and criteria for this will be discussed later.

Hadamard's theorem factorizes the Bargmann function (and more generally all entire functions of finite order ρ) as

$$f_{\mathrm{E}}(z) = z^m \prod_{N=1}^{\infty} E(\zeta_N, p) \exp[Q_q(z)], \tag{133}$$

where E are the Weierstrass factors given by

$$E(\zeta, 0) = \left(1 - \frac{z}{\zeta}\right), \qquad E(\zeta, p) = \left(1 - \frac{z}{\zeta}\right) \exp\left[\frac{z}{\zeta} + \frac{z^2}{2\zeta^2} + \cdots + \frac{z^p}{p\zeta^p}\right]; \tag{134}$$

ζ_N are the zeros of this function. m is the multiplicity of the zero at the origin. $Q_q(z)$ is a polynomial of degree q and p is a non-negative integer. The maximum of (p, q) is called genus of $f(z)$ and does not exceed the order ρ. For Bargmann functions $\rho \leqslant 2$ and therefore p and q can only take the values 0, 1, 2. We stress that Hadamard's theorem factorizes general entire functions of finite order, but only those with growth smaller than ($\rho = 2, \sigma = 1/2$) are acceptable as Bargmann functions.

We express the state of equation (133), in a more physical language [34], in terms of kets and creation operators. Equations (95) and (97) show that with appropriate normalization, $\exp[Q_0(z)]$, $\exp[Q_1(z)]$ and $\exp[Q_2(z)]$ are Bargmann functions for the vacuum, a coherent state and a squeezed state, which in this section we denote as $|Q_0\rangle$, $|Q_1\rangle$ and $|Q_2\rangle$, respectively. On these states we act with the Weierstrass factors, which according to equation (99) we rewrite in terms of creation operators:

$$\hat{E}(\zeta_N, 0) = 1 - \frac{a^\dagger}{\zeta_N}, \tag{135}$$

$$\hat{E}(\zeta_N, 1) = \left[1 - \frac{a^\dagger}{\zeta_N}\right] \exp\left[\frac{a^\dagger}{\zeta_N}\right], \tag{136}$$

$$\hat{E}(\zeta_N, 2) = \left[1 - \frac{a^\dagger}{\zeta_N}\right] \exp\left[\frac{a^\dagger}{\zeta_N} + \frac{a^{\dagger 2}}{2\zeta_N^2}\right]. \tag{137}$$

z^m are the operators $a^{\dagger m}$. All these operators commute with each other and therefore the state with the Bargmann function of equation (133) is

$$|f\rangle = (a^\dagger)^m \prod_{N=1}^{\infty} \hat{E}(\zeta_N, p)|Q_q\rangle, \tag{138}$$

where as we explained $|Q_q\rangle$ is the vacuum for $q = 0$, a coherent state for $q = 1$ and a squeezed state for $q = 2$.

As an example, we consider the following superposition of two coherent states ('Schrodinger cat'):

$$|g\rangle = \mathcal{N}[|\mathrm{i}\rangle_{\mathrm{c}} - |-\mathrm{i}\rangle_{\mathrm{c}}] \to f_{\mathrm{E}}(z) = 2\mathrm{i}\mathcal{N}\,\mathrm{e}^{-1/2}\sin z, \tag{139}$$

where $\mathcal{N}$ is a normalization constant. The Hadamard factorization is in this case

$$\sin z = z\prod_{N=1}^{\infty} E(\zeta_N, 0) = z\prod_{N=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 N^2}\right), \qquad \zeta_{2N-1} = -N\pi, \qquad \zeta_{2N} = N\pi. \tag{140}$$

This implies that all coherent states $|K\pi\rangle_{\mathrm{c}}$, where K is any integer, are orthogonal to the state $|g\rangle$. We can easily check that this is indeed the case.

A quantum state evolves in time, and the corresponding zeros also evolve in time. In order to give an example of this, we consider a system described with the Hamiltonian

$$H = \lambda a + \lambda^* a^\dagger. \tag{141}$$

We assume that at $t = 0$ the system is in the state $|f\rangle$, with Bargmann function $f_{\mathrm{E}}(z)$. In the Bargmann representation, the evolution operator is

$$\exp[\mathrm{i}Ht] = \exp[\mathrm{i}t(\lambda^* z + \lambda\partial_z)] = \exp\left(-\tfrac{1}{2}|\lambda|^2 t^2\right)\exp(\mathrm{i}t\lambda^* z)\exp(\mathrm{i}t\lambda\partial_z). \tag{142}$$

Acting on $f_{\mathrm{E}}(z)$ factorized as in equation (133), we easily see that the zeros evolve in time as

$$\zeta_n(t) = \zeta_n - \mathrm{i}t\lambda. \tag{143}$$

In the case of more complicated Hamiltonians, the study of the motion of the zeros can provide a deeper insight about the system [129]. We stress, however, that knowledge of its zeros does not define uniquely the Bargmann function and the corresponding quantum state. Hadamard's theorem shows clearly that there may be exponential factors in the Bargmann function, with no zeros.

There has been work in the literature on the 'multiphoton squeezing operator' defined as

$$S_k(z) = \exp[z(a^\dagger)^k - z^* a^k], \tag{144}$$

where $k \geqslant 3$. In the special case $k = 2$, this reduces to the squeezing operator of equation (96). It has been shown in [130] that $\langle 0|S_k(z)|0\rangle$ diverges for $k \geqslant 3$, and therefore the states $S_k(z)|0\rangle$ are not normalizable and they do not belong in the Hilbert space. In the approach developed in this section, such states are of the type given in equation (138) with $p \geqslant 3$ and they are not acceptable because in general their growth has order $\rho > 2$ which implies that they are not normalizable.

When confronted with divergences, physicists usually use various techniques (some based on analytic continuation) in order to get finite results and in the present context this has been studied in [131] using Pade approximants. Alternative approaches which have no difficulties with the growth and convergence use appropriate k-photon operators which are different from a^k and $(a^\dagger)^k$:

$$A = \sum_N \lambda_N |N\rangle_{n\,n}\langle N+k|, \qquad A^\dagger = \sum_N \lambda_N |N+k\rangle_{n\,n}\langle N|. \tag{145}$$

For example, reference [92] has used the Brandt–Greenberg operators [132], and reference [93] has introduced some other operators motivated from conformal field theory, which form a Lie algebra.

In a more general context, it should be pointed out that in many cases physicists enlarge the Hilbert space and use some technique to get a finite result for the scalar product. An example is the coupled cluster method in many-body theory and quantum chemistry [133] (see also [74]) which uses exponentials of a^k and $(a^\dagger)^k$ with $k \geqslant 3$. In these cases, extra care is required in order to define explicitly and clearly the 'size' of the Hilbert space and the scalar product in it, and the language of analytic functions and their growth is very useful in this direction.

The difficulties associated with the multiphoton squeezing operator of equation (144) is one example where the analytic techniques discussed in this review provide a deeper insight.

5.4. Completeness of sequences of coherent states

We first consider a sequence of complex numbers $\{\zeta_N\}$ which has a limit w. The corresponding set of coherent states $\{|\zeta_N\rangle_c\}$ is a small subset of the full set of all coherent states, but it is overcomplete. Indeed, if it was undercomplete there would exist a state which is orthogonal to all $\{|\zeta_N\rangle_c\}$, but this would mean that the corresponding Bargmann function has all the points $\{\zeta_N\}$ as zeros. This is not possible, because the zeros of analytic functions are isolated from each other and they cannot have a limit w. This proves that the set of coherent states $\{|\zeta_N\rangle_c\}$ is at least complete. In fact, it is overcomplete, because the same argument is true even if we omit from the sequence any finite number of points.

We next consider a sequence $\{\zeta_N\}$ of the type described in equation (127). We will prove that the corresponding set of coherent states $\{|\zeta_N\rangle_c\}$ is overcomplete when the density of this sequence is larger than $(\eta = 2, \delta = 1)$, and undercomplete when the density is smaller than $(\eta = 2, \delta = 1)$. In the borderline case that the density of the sequence is equal to $(\eta = 2, \delta = 1)$, the corresponding set of coherent states might be overcomplete or undercomplete.

We first discuss the case where the density of the sequence is larger than (2, 1). If the corresponding sequence of coherent states is undercomplete, there exists a state $|f\rangle$ which is orthogonal to all $\{|\zeta_N\rangle_c\}$. Its Bargmann function $f_E(z)$ has all the points $\{\zeta_N\}$ as zeros. Taking into account equation (132), we conclude that this Bargmann function would have growth greater than (2, 1/2), which is not possible. This proves that the set of coherent states $\{|\zeta_N\rangle_c\}$ is at least complete. In fact, it is overcomplete, because the same argument is true even if we omit from the sequence any finite number of points.

We next consider the case where the density of the sequence $\{\zeta_N\}$ is smaller than (2, 1). In this case, the corresponding analytic function has growth less than (2, 1/2) and is acceptable as a Bargmann function. We then use Hadamard's theorem in equation (138) to construct explicitly a state $|f\rangle$ which is orthogonal to all coherent states $\{|\zeta_N\rangle_c\}$. This shows clearly the undercomplete nature of $\{|\zeta_N\rangle_c\}$.

As an example, we consider the sequence of complex numbers given in equation (131). The corresponding set of coherent states is overcomplete for $\eta_0 = 2.1$ and undercomplete for $\eta_0 = 1.9$.

Work that studies the completeness of a random set of coherent states has been presented in [134]. This is not discussed here.

5.5. *The von Neumann lattice of coherent states*

An important example for the application of the above general results is the von Neumann lattice. It consists of the points $\{z_{MN} = S^{1/2}(M + \mathrm{i}N)\}$ where M, N are integers and S is the area of the lattice cell. It is easily seen that its density is $(\eta = 2, \delta = S/\pi)$. According to our general result, the corresponding von Neumann lattice of coherent states is overcomplete when $S < \pi$ and undercomplete when $S > \pi$.

We construct explicitly a state $|f\rangle$ which is orthogonal to all coherent states in a von Neumann lattice with $S = 4$. A well-known function with zeros at the points $2(M + \mathrm{i}N)$ is the Weierstrass sigma function:

$$\sigma(z|1, \mathrm{i}) = z \prod E(\zeta_{MN}, 2), \qquad \zeta_{MN} = 2(M + \mathrm{i}N), \tag{146}$$

where the products includes all $(M, N) \neq (0, 0)$ and

$$E(\zeta_{MN}, 2) = \left[1 - \frac{z}{\zeta_{MN}}\right] \exp\left[\frac{z}{\zeta_{MN}} + \frac{z^2}{2\zeta_{MN}^2}\right]. \tag{147}$$

The state with Bargmann function $\mathcal{N}\sigma(z|1, \mathrm{i})$, where $\mathcal{N}$ is appropriate normalization factor, is orthogonal to all coherent states in the von Neumann lattice $\{|2(M + \mathrm{i}N)\rangle_c\}$.

We note that in the von Neumann lattice we have a uniform distribution of points in the complex plane, but this is certainly *not* a requirement for our general result, and the example of equation (131) with an irregular distribution of points in the complex plane was given precisely in order to emphasize this statement.

In the 'borderline case' $S = \pi$, the von Neumann lattice of coherent states is known to be overcomplete by one state [1]. We next consider the same lattice with a finite number of points added to it, or a finite number of points subtracted from it. We have explained that this does not change its density (η, δ). But clearly this 'enlarged' or 'truncated' von Neumann lattice is overcomplete or undercomplete. This exemplifies our earlier statement that in the borderline case the density of the sequence is equal to $(\eta = 2, \delta = 1)$, the corresponding set of coherent states might be overcomplete or undercomplete.

In the von Neumann lattice with $S = \pi$, i.e. $\{z_{MN} = \pi^{1/2}(M + \mathrm{i}N)\}$, the corresponding displacement operators commute:

$$D(z_{MN})D(z_{K\Lambda}) = D(z_{K\Lambda})D(z_{MN}). \tag{148}$$

The common eigenvectors of these commuting operators have been studied in [1, 135]. They involve theta functions and they are not normalizable; therefore, they do not belong to the usual harmonic oscillator $\mathcal{H}$, but belong to an extended space. We do not discuss this approach here, but we will introduce an analytic representation that involves theta functions later.

We note that the von Neumann lattice has been studied in a signal processing context by Gabor [136]. This work has developed into the time–frequency approach which is an important tool in signal analysis [137]. Most of the ideas described in this review in a quantum-mechanical language are also applicable to signal analysis with a different terminology.

6. Euclidean contour representation in the complex plane

6.1. States

In the Euclidean contour formalism, the arbitrary ket state $|f\rangle$ of equation (88) and the corresponding bra state $\langle f|$ are represented as [45–49]

$$\begin{aligned}|f\rangle &\to f_{\mathrm{Ek}}(z) = \sum_N f_N (N!)^{-1/2} z^N,\\ \langle f| &\to f_{\mathrm{Eb}}(z) = \sum_N f_N^* (N!)^{1/2} z^{-N-1},\end{aligned} \tag{149}$$

where the indices 'Ek' and 'Eb' refer to 'Euclidean and ket' and 'Euclidean and bra', respectively. The ket function $f_{\mathrm{Ek}}(z)$ is the same as the Bargmann function $f_{\mathrm{E}}(z)$ and is analytic in the complex plane with growth smaller than $(\rho = 2, \sigma = 1/2)$. The bra function $f_{\mathrm{Eb}}(z)$ can be problematic due to convergence difficulties.

Given a series $\sum u_N$, we define

$$q = \lim_{N\to\infty} |u_N|^{1/N}. \tag{150}$$

The series converges absolutely when $q < 1$ and diverges when $q > 1$. For the series associated with $f_{\mathrm{Eb}}(z)$, we find

$$q = \frac{1}{|z|} \lim_{N\to\infty} [(N!)^{1/2}|f_N|]^{1/N}. \tag{151}$$

We call $\mathcal{H}(\lambda)$ (with $\lambda \geqslant 0$) the space

$$\mathcal{H}(\lambda) = \{|f\rangle : \lim_{N\to\infty} [(N!)^{1/2}|f_N|]^{1/N} \leqslant \lambda\}. \tag{152}$$

When $\lambda < \lambda'$, then the space $\mathcal{H}(\lambda)$ is a subspace of $\mathcal{H}(\lambda')$. For states in $\mathcal{H}(\lambda)$, the series associated with $f_{\mathrm{Eb}}(z)$ converges in the annulus $\mathcal{T}(\lambda)$ (defined in equation (1)). We next show that the order of the growth of the ket functions $f_{\mathrm{Ek}}(z)$ corresponding to the states in the space $\mathcal{H}(\lambda)$ is $\rho \leqslant 1$, for any positive λ. Heuristically, we easily see this from the relation of equation (152) which gives $|f_N| < \lambda^N (N!)^{-1/2}$ for large N, and this leads to $|f_{\mathrm{Ek}}(z)| < \mathrm{e}^{\lambda|z|}$. More formally, we express the ratio entering the limit of equation (123) as

$$\frac{N \ln N}{-\ln[(N!)^{-1/2}|f_N|]} = \frac{N \ln N}{-\ln[(N!)^{1/2}|f_N|] + \ln(N!)} \leqslant \frac{N \ln N}{-N \ln\lambda + \ln(N!)}. \tag{153}$$

In the limit $N \to \infty$, according to Stirling's formula $\ln(N!)$ behaves like $N \ln N$ and therefore the order of the growth of the functions $f_{\mathrm{Ek}}(z)$ corresponding to the states in the space $\mathcal{H}(\lambda)$ is $\rho \leqslant 1$. Therefore, the space $\mathcal{H}(\lambda)$ is *not* dense in the full Hilbert space $\mathcal{H}$, for any λ .

The states which have order of the growth of the ket function $f_{\mathrm{Ek}}(z)$ in the region $1 < \rho \leqslant 2$ need to be considered on an individual basis. For many of them, the series related to the bra function $f_{\mathrm{Eb}}(z)$ will converge in a small region and analytic continuation might be used to define it in larger regions. In any case, physics abounds with techniques dealing with divergent series (and integrals) which might be useful in dealing with these states in the context of the Euclidean contour formalism. Notwithstanding this weakness, the method has been studied extensively in the literature [45–51] and used in [48, 51] to expand an arbitrary state in terms of coherent states on a circle and in [49] to provide a quantum formalism for negative temperatures.

The contour integrals below are for states in one of the spaces $\mathcal{H}(\lambda)$ and involve integration along an anticlockwise contour ℓ within the annulus $\mathcal{T}(\lambda)$. Clearly, all singularities of $f_{\mathrm{Eb}}(z)$ are enclosed within the contour ℓ and the contour integrals are rigorously defined. The scalar product of two states is given by

$$\langle f|g\rangle = \oint_{\ell\in\mathcal{T}(\lambda)} \frac{\mathrm{d}z}{2\pi\mathrm{i}} f_{\mathrm{Eb}}(z) g_{\mathrm{Ek}}(z). \tag{154}$$

The following transforms connect $f_{\mathrm{Eb}}(z)$ with the $f_{\mathrm{Ek}}(\zeta)$:

$$\oint_{\ell\in\mathcal{T}(\lambda)} \frac{\mathrm{d}z}{2\pi\mathrm{i}} f_{\mathrm{Eb}}(z)\exp(\zeta^* z) = [f_{\mathrm{Ek}}(\zeta)]^*. \tag{155}$$

The inverse formula is the Laplace transform,

$$f_{\mathrm{Eb}}(z) = \frac{1}{z}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t}\left[f_{\mathrm{Ek}}\left(\frac{t}{z^*}\right)\right]^*. \tag{156}$$

It is interesting that for the states in $\mathcal{H}(\lambda)$, the order of the growth of the ket functions is at most 1 and we get absolute convergence of this integral (at least for large enough $|z|$). For functions with growth greater than 1, we might get convergence in a limit region of the values of z. Not surprisingly, the requirements for the existence of $f_{\mathrm{Eb}}(z)$ are the same either through the series in equation (149) or through the integral in equation (156).

As an example, we consider the coherent state $|w\rangle_{\mathrm{c}}$ and we get

$$f_{\mathrm{Ek}}(z) = \exp\left(-\tfrac{1}{2}|w|^2 + wz\right), \tag{157}$$

$$f_{\mathrm{Eb}}(z) = \exp\left(-\tfrac{1}{2}|w|^2\right)(z-w^*)^{-1}, \qquad |z| > |w|. \tag{158}$$

6.2. Operators

An arbitrary operator $\mathcal{U}$ with matrix elements $\mathcal{U}_{MN} = {}_n\langle M|\mathcal{U}|N\rangle_n$ is represented by the following kernel:

$$\mathcal{C}_{\mathrm{E}}(z_1, z_2; \mathcal{U}) = \sum_{M,N=0}^{\infty} \mathcal{U}_{MN}\left(\frac{N!}{M!}\right)^{1/2}\frac{z_1^M}{z_2^{N+1}}. \tag{159}$$

This function is useful if the double series converges for $z_1 \in \mathcal{S}$ and for $z_2 \in \mathcal{T}$ where $\mathcal{S}, \mathcal{T}$ are regions defined in equation (1) with radii that depend on the operator.

We consider the following examples:

$$\begin{aligned}
\mathcal{C}_{\mathrm{E}}(z_1, z_2; \mathbf{1}) &= (z_2 - z_1)^{-1}, & |z_2| &> |z_1|,\\
\mathcal{C}_{\mathrm{E}}(z_1, z_2; a) &= (z_2 - z_1)^{-2}, & |z_2| &> |z_1|,\\
\mathcal{C}_{\mathrm{E}}(z_1, z_2; a^\dagger) &= z_1(z_2 - z_1)^{-1}, & |z_2| &> |z_1|,\\
\mathcal{C}_{\mathrm{E}}(z_1, z_2; a^\dagger a) &= z_1(z_2 - z_1)^{-2}, & |z_2| &> |z_1|.
\end{aligned} \tag{160}$$

In all these examples, $z_1 \in \mathcal{S}(r)$ and $z_2 \in \mathcal{T}(r)$ for arbitrary r.

If $|f\rangle$ belongs in $\mathcal{H}(\lambda)$ then the ket state $|g\rangle = \mathcal{U}|f\rangle$ and the bra state $\langle g| = \langle f|\mathcal{U}^{\dagger}$ are represented as follows:

$$\begin{aligned} g_{\mathrm{Ek}}(z_1) &= \oint_{\ell\in\mathcal{T}(r)} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \mathcal{C}_{\mathrm{E}}(z_1, z_2; \mathcal{U}) f_{\mathrm{Ek}}(z_2), \\ g_{\mathrm{Eb}}(z_2) &= \oint_{\ell\in\mathcal{T}(\lambda)\cap\mathcal{S}(r)} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} f_{\mathrm{Eb}}(z_1)\mathcal{C}_{\mathrm{E}}(z_1, z_2; \mathcal{U}^{\dagger}). \end{aligned} \tag{161}$$

In the first integral $z_1 \in \mathcal{S}(r)$ and $z_2 \in \mathcal{T}(r)$. Since $g_{\mathrm{Ek}}(z_1)$ is an entire function, through analytic continuation it is defined in the whole complex plane. In the second integral, $z_1 \in \mathcal{T}(\lambda) \cap \mathcal{S}(r)$ (and we have to choose $r > \lambda$) and $z_2 \in \mathcal{T}(r)$.

We use equations (161) with any of the kernels in equation (160). The first integral involves integration over z_2 and the singularity of $\mathcal{C}_{\mathrm{E}}(z_1, z_2; \mathcal{U})$ at $z_2 = z_1$ is inside the contour ℓ. The second integral involves integration over z_1 and the singularity of $\mathcal{C}_{\mathrm{E}}(z_1, z_2; \mathcal{U})$ at $z_1 = z_2$ is outside the contour ℓ; it is only the singularities of $f_{\mathrm{Eb}}(z_1)$ that contribute to this integral. For example, the reproducing kernel relation that involves the unit operator acting on a state $|f\rangle$ is in the present formalism the familiar relation from the theory of analytic functions:

$$\oint_{\ell\in\mathcal{S}_2} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \frac{f_{\mathrm{Ek}}(z_2)}{z_2 - z_1} = f_{\mathrm{Ek}}(z_1). \tag{162}$$

We also prove that the eigenkets of the number operators are indeed the number eigenstates:

$$\oint_{\ell\in\mathcal{S}_2} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \frac{z_1}{(z_1 - z_2)^2} z_2^N (N!)^{-1/2} = N z_1^N (N!)^{-1/2} \tag{163}$$

and that the eigenkets of the annihilation operator are indeed the coherent states:

$$\oint_{\ell\in\mathcal{S}_2} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \frac{1}{(z_1 - z_2)^2} \exp\left(-\frac{1}{2}|w|^2 + wz_2\right) = w \exp\left(-\frac{1}{2}|w|^2 + wz_1\right). \tag{164}$$

We have explained earlier that if $\{\zeta_N\}$ is a sequence which has a limit w, then the coherent states $\{|\zeta_N\rangle_{\mathrm{c}}\}$ form an overcomplete set. Therefore, the coherent states in a contour ℓ form a highly overcomplete set of states. This means that, in principle, an arbitrary state can be expanded in many different ways in terms of these coherent states. However, in practice, it is not easy to find such an expansion. The Euclidean contour formalism provides such an expansion. An arbitrary state $|f\rangle \in \mathcal{H}(\lambda)$ can be written as

$$|f\rangle = \oint_{\ell\in\mathcal{R}(\lambda)} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \exp\left(\frac{|z|^2}{2}\right) a(z)|z\rangle_{\mathrm{c}}, \qquad a(z) = \sum_N \frac{f_N (N!)^{1/2}}{z^{N+1}} = [f_{\mathrm{Eb}}(z^*; k)]^*. \tag{165}$$

This has been studied and used in a more applied context in [48].

Part II: hyperbolic analytic representations

7. Basic $SU(1, 1)$ formalism

7.1. $SU(1,1)$ generators and their polar decomposition

The unitary irreducible representations of $SU(1, 1)$ have been studied in detail in [16–19] (see also [138, 139]). They are classified into the continuous series, the discrete series and the supplementary series. We study mainly the *discrete series* labelled with k, which takes the values $= 1/2, 1, 3/2, \ldots$. In a few cases, we will use the $k = 1/4$ and $k = 3/4$ representations which belong in the continuous series, and we will make this clear in the text.

We consider the unitary irreducible representation labelled with k and the generators K_0, K_+, K_- which obey the commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_-, K_+] = 2K_0. \tag{166}$$

The Casimir operator in this representation is

$$K^2 = K_0^2 - \tfrac{1}{2}(K_+K_- + K_-K_+) = k(k-1)\mathbf{1}. \tag{167}$$

We also consider the '$SU(1,1)$ number states' $|N;k\rangle_n$ defined as

$$\begin{aligned}
K^2|N;k\rangle_n &= k(k-1)|N;k\rangle_n,\\
K_0|N;k\rangle_n &= (k+N)|N;k\rangle_n,\\
K_-|N;k\rangle_n &= [N(N+2k-1)]^{1/2}|N-1;k\rangle_n,\\
K_+|N;k\rangle_n &= [(N+1)(N+2k)]^{1/2}|N+1;k\rangle_n,
\end{aligned} \tag{168}$$

where $N = 0, 1, 2, \ldots$. The term number states indicates here that they are eigenstates of K_0, labelled with the non-negative integer N. The states $|N;k\rangle_n$ (with fixed k) span an infinite-dimensional Hilbert space which we call $\mathcal{H}_k$.

We next introduce a polar decomposition of the 'Cartesian operators' K_+ and K_- [142]. This is similar to the polar decomposition of the creation and annihilation operators $a^\dagger, a$ introduced earlier in equation (20). We consider the 'polar operators'

$$K_\mathrm{r} = (K_+K_-)^{1/2}, \qquad E_+ = E_-^\dagger = \sum_{N=0}^{\infty} |N+1;k\rangle_{n\,n}\langle N;k|, \tag{169}$$

where K_r is the radial operator and E_+, E_- are '$SU(1,1)$ exponential of the phase operators'. We can prove that

$$\begin{aligned}
K_+ &= K_\mathrm{r}E_+, & K_- &= E_-K_\mathrm{r},\\
E_-E_+ &= \mathbf{1}, & E_+E_- &= \mathbf{1} - |0;k\rangle_{n\,n}\langle 0;k|,\\
K_\mathrm{r}^2 &= K_0^2 - K_0 - k(k-1)\mathbf{1}, & [K_\mathrm{r}, K_0] &= 0.
\end{aligned} \tag{170}$$

E_+ is an isometric operator, but it is *not* a unitary operator.

7.2. $SU(1,1)$ *transformations*

We consider the unitary $SU(1,1)$ operators

$$\begin{aligned}
&S(r,\theta,\lambda;k) = \exp\left[-\tfrac{1}{2}r\,\mathrm{e}^{-\mathrm{i}\theta}K_+ + \tfrac{1}{2}r\,\mathrm{e}^{\mathrm{i}\theta}K_-\right]\exp(\mathrm{i}\lambda K_0), \qquad r \geqslant 0,\\
&0 \leqslant \theta < 2\pi, \qquad 0 \leqslant \lambda < 2\pi, \qquad k = 1/2, 1, 3/2, \ldots.
\end{aligned} \tag{171}$$

We will also use the following notation:

$$S(z;k) = S(r,\theta,0;k), \qquad z = -\mathrm{e}^{-\mathrm{i}\theta}\tanh\frac{r}{2}, \tag{172}$$

where z is in the unit disc D. The product of two of these operators is given by

$$S(z_1;k)S(z_2;k) = S(w;k)\exp(-\mathrm{i}\phi K_0), \tag{173}$$

where

$$w = \frac{z_1+z_2}{1+z_1^*z_2}, \qquad \phi = 2\arg(1+z_1^*z_2). \tag{174}$$

This is the analogue of equation (33) in the Euclidean case.

It can be shown [1] that

$$\begin{aligned} &S(z;k)K_0[S(z;k)]^\dagger = K_{\eta(z)},\\ &K_{\eta(z)} = \eta_0 K_0 - \tfrac{1}{2}(\eta_- K_+ + \eta_+ K_-),\\ &\eta_0 = \frac{1+|z|^2}{1-|z|^2}, \qquad \eta_- = \eta_+^* = \frac{\mathrm{i}2z}{1-|z|^2}. \end{aligned} \tag{175}$$

An important property of the operators $S(z;k)$ is the 'generalized resolution of the identity'. Let $\mathcal{U}$ be an arbitrary trace class operator acting on $\mathcal{H}_k$. We show that

$$\int_D \mathrm{d}\mu_{\mathrm{H}}(z) S(z;k)\frac{\mathcal{U}}{\operatorname{Tr}\mathcal{U}}[S(z;k)]^\dagger = \mathbf{1}. \tag{176}$$

Perhaps, the easiest way to prove this is to use the P representation of the operator which is introduced later. We act with the operators $S(z;k)$ and $[S(z;k)]^\dagger$ on both sides of the operator $\mathcal{U}$ in equation (191) and prove equation (176). In the special case that $\mathcal{U} = |0;k\rangle_{n\,n}\langle 0;k|$, this relation becomes the resolution of the identity for the $SU(1,1)$ coherent states which is discussed below.

7.3. $SU(1,1)$ coherent states

$SU(1,1)$ coherent states are defined in the coset space $SU(1,1)/U(1)$ which is the upper sheet of a hyperboloid. This is isomorphic to the unit disc D which is the Poincare model of the Lobachevsky geometry.

$SU(1,1)$ coherent states [1] in the unit disc are defined as

$$|z;k\rangle_{\mathrm{c}} = (1-|z|^2)^k \sum_{N=0}^{\infty} d_{\mathrm{H}}(N;k) z^N |N;k\rangle_n, \qquad |z| < 1, \tag{177}$$

where

$$d_{\mathrm{H}}(N;k) = \left[\frac{\Gamma(N+2k)}{\Gamma(N+1)\Gamma(2k)}\right]^{1/2}. \tag{178}$$

An alternative equivalent definition is

$$|z;k\rangle_{\mathrm{c}} = S(z;k)|0;k\rangle_n. \tag{179}$$

The equivalence can be shown using the relation

$$\begin{aligned} &\exp\left[-\frac{1}{2}r\,\mathrm{e}^{-\mathrm{i}\theta}K_+ + \frac{1}{2}r\,\mathrm{e}^{\mathrm{i}\theta}K_-\right] = \exp[zK_+]\exp[\tau K_0]\exp[-z^*K_-],\\ &z = -\mathrm{e}^{-\mathrm{i}\theta}\tanh\frac{r}{2}, \qquad \tau = \ln(1-|z|^2). \end{aligned} \tag{180}$$

The Baker–Campbell–Hausdorff relations for $SU(1,1)$ have been discussed in [140] using the general approach of [141].

Use of equation (175) leads to a third definition of $SU(1,1)$ coherent states as eigenstates of the Hermitian operator K_η:

$$K_{\eta(z)}|z;k\rangle_{\mathrm{c}} = k|z;k\rangle_{\mathrm{c}}. \tag{181}$$

These states are coherent states in the sense that if we act with any of the operators $S(r,\theta,\lambda;k)$ on any of these coherent states, we get another coherent state. This is because $S(r,\theta,\lambda;k)$ form a group (a representation of the $SU(1,1)$ group).

The probability distribution

$$|{}_n\langle N;k|z;k\rangle_{\mathrm{c}}|^2 = (1-|z|^2)^{2k}\frac{\Gamma(N+2k)}{\Gamma(N+1)\Gamma(2k)}|z|^{2N} \tag{182}$$

is a negative binomial distribution.

The overlap between two coherent states is given by

$$ {}_{\mathrm{c}}\langle z_1; k | z_2; k\rangle_{\mathrm{c}} = \frac{(1-|z_1|^2)^k (1-|z_2|^2)^k}{(1-z_1^* z_2)^{2k}}. \tag{183} $$

For later use, we introduce

$$ \mathcal{G}_{\mathrm{H}}(z_1, z_2; k) \equiv |{}_{\mathrm{c}}\langle z_1; k|z_2; k\rangle_{\mathrm{c}}|^2 = \left[\frac{(1-|z_1|^2)(1-|z_2|^2)}{|1-z_1 z_2^*|^2}\right]^{2k}. \tag{184} $$

There is a connection between this and the distance $\delta_{\mathrm{H}}(z_1, z_2)$ between the points z_1 and z_2 in Lobachevsky geometry (unit disc):

$$ \mathcal{G}_{\mathrm{H}}(z_1, z_2; k) = \{1 - [\tanh \delta_{\mathrm{H}}(z_1, z_2)]^2\}^k. \tag{185} $$

The resolution of the identity of these states is given by

$$ \frac{2k-1}{\pi} \int_D |z; k\rangle_{\mathrm{c}}\, {}_{\mathrm{c}}\langle z; k| \, \mathrm{d}\mu_{\mathrm{H}}(z) = \mathbf{1}. \tag{186} $$

This shows that the set of all $SU(1,1)$ coherent states is at least complete. In fact, it is highly overcomplete because we will prove below that small subsets of these coherent states are also overcomplete.

We next mention some special features of the case $k = 1/2$. In this case, equation (177) reduces to

$$ \left| z; \frac{1}{2} \right\rangle_{\mathrm{c}} = (1-|z|^2)^{1/2} \sum_{N=0}^{\infty} z^N \left| N; \frac{1}{2} \right\rangle_n \tag{187} $$

and the corresponding probability distribution

$$ \left| {}_n\left\langle N; \frac{1}{2} \middle| z; \frac{1}{2} \right\rangle_{\mathrm{c}} \right|^2 = (1-|z|^2)|z|^{2N} \tag{188} $$

is a thermal distribution. The resolution of the identity is given as the one-sided limit $k \to 1/2 + \epsilon$ of relation (186):

$$ \lim_{k \to 1/2+\epsilon} \frac{2k-1}{\pi} \int_D |z; k\rangle_{\mathrm{c}}\, {}_{\mathrm{c}}\langle z; k| \, \mathrm{d}\mu_{\mathrm{H}}(z) = \mathbf{1}. \tag{189} $$

We note that this integral diverges for $k < 1/2$.

7.4. P, Q and Wigner functions

The Q and P functions of a bounded operator $\mathcal{U}$ are defined as

$$ Q_{\mathrm{H}}(z; \mathcal{U}) = {}_{\mathrm{c}}\langle z; k|\mathcal{U}|z; k\rangle_{\mathrm{c}}, \tag{190} $$

$$ \mathcal{U} = \frac{2k-1}{\pi} \int_D \mathrm{d}\mu_{\mathrm{H}}(z) P_{\mathrm{H}}(z; \mathcal{U}) |z; k\rangle_{\mathrm{c}}\, {}_{\mathrm{c}}\langle z; k|, \tag{191} $$

where $|z| < 1$. We combine these two relations and prove that for $k > 1/2$:

$$ Q_{\mathrm{H}}(z; \mathcal{U}) = \frac{2k-1}{\pi} \int_D \mathrm{d}\mu_{\mathrm{H}}(w)\, \mathcal{G}_{\mathrm{H}}(z, w; k) P_{\mathrm{H}}(w; \mathcal{U}). \tag{192} $$

The trace of an operator $\mathcal{U}$ is given by

$$ \mathrm{Tr}\, \mathcal{U} = \frac{2k-1}{\pi} \int_D \mathrm{d}\mu_{\mathrm{H}}(z) Q_{\mathrm{H}}(z; \mathcal{U}) = \frac{2k-1}{\pi} \int_D \mathrm{d}\mu_{\mathrm{H}}(z) P_{\mathrm{H}}(z; \mathcal{U}). \tag{193} $$

The trace of the product $\mathcal{U}_1\mathcal{U}_2$ of two operators can be written as

$$\mathrm{Tr}(\mathcal{U}_1\mathcal{U}_2) = \frac{2k-1}{\pi}\int_D \mathrm{d}\mu_{\mathrm{H}}(z) P_{\mathrm{H}}(z;\mathcal{U}_1) Q_{\mathrm{H}}(z;\mathcal{U}_2). \tag{194}$$

We next follow Berezin [30] and define the displaced parity operator and Wigner functions. The parity operator around the origin $\mathcal{R}_0$ is defined as

$$\mathcal{R}_0 = \sum_N (-1)^N |N;k\rangle\langle N;k|. \tag{195}$$

The displaced parity operator around a point w is defined as

$$\mathcal{R}(w) = S(r,\theta,\lambda;k)\mathcal{R}_0[S(r,\theta,\lambda;k)]^\dagger, \qquad w = -\tanh\left(\frac{r}{2}\right)\mathrm{e}^{-\mathrm{i}(\lambda-\theta)}, \tag{196}$$

where the operators $S(r,\theta,\lambda;k)$ have been defined in equation (171). Acting on coherent states with it, we get

$$\mathcal{R}(w)|z;k\rangle_{\mathrm{c}} = |\zeta;k\rangle_{\mathrm{c}}, \qquad \zeta = \frac{-(1+|w|^2)z + 2w}{-2w^*z + (1+|w|^2)}. \tag{197}$$

We note that the point ζ is the reflection of the point z around w in the unit disc (Lobachevsky geometry). The Q function of the operator $\mathcal{R}(w)$ is

$$Q_{\mathrm{H}}(z;\mathcal{R}(w)) = \mathcal{G}_{\mathrm{H}}(z,w;k)\left(1+\left|\frac{z-w}{1-z^*w}\right|^2\right)^{-2k}. \tag{198}$$

For an arbitrary operator $\mathcal{U}$, we define the Wigner function as

$$W_{\mathrm{H}}(z;\mathcal{U}) = \mathrm{Tr}(\mathcal{U}\mathcal{R}(z)). \tag{199}$$

Using equation (194) we can show that the Wigner function of an operator is related to its P function through the relation

$$W_{\mathrm{H}}(z;\mathcal{U}) = \frac{2k-1}{\pi}\int_D \mathrm{d}\mu_{\mathrm{H}}(w) P_{\mathrm{H}}(w;\mathcal{U})\mathcal{G}_{\mathrm{H}}(z,w;k)\left(1+\left|\frac{z-w}{1-z^*w}\right|^2\right)^{-2k}. \tag{200}$$

7.5. $SU(1,1)$ phase states

We introduce phase states in the space $\mathcal{H}_k$ [142]. The phase states in the various $\mathcal{H}_k$ behave in isomorphic way, and for this reason we omit the index 'k' in their notation.

We call $SU(1,1)$ phase states the eigenstates of the $SU(1,1)$ exponential of the phase operators E_-:

$$\begin{aligned} &E_-|z\rangle_{\mathrm{ph}} = z|z\rangle_{\mathrm{ph}}, \qquad |z| < 1, \\ &|z\rangle_{\mathrm{ph}} = (1-|z|^2)^{1/2}\sum_{N=0}^{\infty} z^N |N;k\rangle_n. \end{aligned} \tag{201}$$

The index 'ph' stands for phase states. We also introduce the bounded operator

$$(\mathbf{1} - zE_+)^{-1} \equiv \sum_{N=0}^{\infty}(zE_+)^N, \qquad |z| < 1, \tag{202}$$

and write the phase states as

$$|z\rangle_{\mathrm{ph}} = (1-|z|^2)^{1/2}(\mathbf{1}-zE_+)^{-1}|0;k\rangle_n. \tag{203}$$

We note that there are no eigenkets of the operator E_+.

$SU(1, 1)$ phase states are different from the $SU(1, 1)$ coherent states. Only in the special case $k = 1/2$, the phase states are identical with the coherent states

$$|z\rangle_{\mathrm{ph}} = \left|z; \tfrac{1}{2}\right\rangle_{\mathrm{c}}. \tag{204}$$

The overlap of two phase states is

$${}_{\mathrm{ph}}\langle\zeta|z\rangle_{\mathrm{ph}} = \frac{(1-|\zeta|^{1/2})^{1/2}(1-|z|^2)^{1/2}}{1-\zeta^* z}. \tag{205}$$

Inversion of equation (201) can be achieved with integration along a circle in the unit disc, with centre at the origin and radius $r < 1$:

$$\begin{aligned} &r^{-N}(1-r^2)^{-1/2}\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\,\mathrm{e}^{-\mathrm{i}N\theta}|r\,\mathrm{e}^{\mathrm{i}\theta}\rangle_{\mathrm{ph}} = |N; k\rangle_n, \qquad N \geqslant 0,\\ &\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\,\mathrm{e}^{-\mathrm{i}N\theta}|r\,\mathrm{e}^{\mathrm{i}\theta}\rangle_{\mathrm{ph}} = 0, \qquad N < 0. \end{aligned} \tag{206}$$

These relations show that the set of phase states is overcomplete in the space $\mathcal{H}_k$. Indeed, using equation (206) we show that an arbitrary normalized state in $\mathcal{H}_k$:

$$|f\rangle = \sum_{N=0}^{\infty} f_N|N, k\rangle_n, \qquad \sum_{N=0}^{\infty}|f_N|^2 = 1 \tag{207}$$

can be expanded in terms of $SU(1, 1)$ phase states as

$$|f\rangle = r^{-N}(1-r^2)^{-k}\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}f(\theta)|r\,\mathrm{e}^{\mathrm{i}\theta}\rangle_{\mathrm{ph}}, \qquad f(\theta) = \sum_{N=0}^{\infty} f_N\,\mathrm{e}^{-\mathrm{i}N\theta}. \tag{208}$$

7.6. The Barut–Girardello states

In the Euclidean case, we get coherent states by acting with the displacement operator $D(z)$ on the vacuum, and these states are eigenstates of the annihilation operator which is non-Hermitian. In the hyperbolic case, we get coherent states by acting with the $SU(1, 1)$ operator $S(r, \theta, 0; k)$ on the lowest state $|0; k\rangle_n$, and these states are eigenstates of the Hermitian operator K_η (equation (181)), but they are not eigenstates of the non-Hermitian operator K_-.

Barut and Girardello [52] (and also [53]) constructed the eigenstates of K_-:

$$\begin{aligned} &|z; k\rangle_{\mathrm{BG}} = \frac{z^{k-1/2}}{[I_{2k-1}(2|z|)]^{1/2}}\sum_{N=0}^{\infty} d_{\mathrm{BG}}(N; k)z^N|N; k\rangle_n,\\ &d_{\mathrm{BG}}(N; k) = [N!\Gamma(n+2k)]^{-1/2},\\ &K_-|z; k\rangle_{\mathrm{BG}} = z|z; k\rangle_{\mathrm{BG}}. \end{aligned} \tag{209}$$

Here z is a complex number and $I_{2k-1}(x)$ is modified Bessel function of the first kind. Barut and Girardello used the term coherent states for these states, but we use this term for the states of equation (177) which are different. We simply refer to them as Barut–Girardello states or as eigenstates of K_-, and we use the index 'BG' in their notation.

The overlap of two of these states is given by

$${}_{\mathrm{BG}}\langle\zeta; k|z; k\rangle_{\mathrm{BG}} = \frac{I_{2k-1}[2(\zeta^* z)^{1/2}]}{[I_{2k-1}(2|z|)I_{2k-1}(2|\zeta|)]^{1/2}}. \tag{210}$$

There is a resolution of the identity in terms of these states

$$\begin{aligned} &\int_C \mathrm{d}\mu_{\mathrm{BG}}(z; k)I_{2k-1}(2|z|)|z; k\rangle_{\mathrm{BG}\ \mathrm{BG}}\langle z, k| = \mathbf{1},\\ &\mathrm{d}\mu_{\mathrm{BG}}(z, k) = \frac{2}{\pi}K_{2k-1}(2|z|)\,\mathrm{d}z_{\mathrm{R}}\,\mathrm{d}z_{\mathrm{I}}, \end{aligned} \tag{211}$$

where $K_{2k-1}(x)$ is modified Bessel function of the second kind. We stress that the integration is over the whole complex plane. The resolution of the identity shows that the set of all Barut–Girardello states is at least complete. In fact, it is highly overcomplete because we will prove below that small subsets of these states are also overcomplete.

For later use, we calculate the overlap between a Barut–Girardello state and an $SU(1,1)$ coherent state

$$ {}_{\mathrm{c}}\langle\zeta;k|z;k\rangle_{\mathrm{BG}} = \frac{z^{k-1/2}(1-|\zeta|^2)^k\exp(\zeta^* z)}{[I_{2k-1}(2|z|)\Gamma(2k)]^{1/2}}, \qquad \zeta\in D, \quad z\in C. \tag{212} $$

8. The $SU(1,1)$ formalism in the harmonic oscillator context

8.1. The $k = 1/4$ and $k = 3/4$ representations and generalized squeezing

We consider the following realization of the $SU(1,1)$ algebra in terms of harmonic oscillator creation and annihilation operators $a^\dagger, a$:

$$ K_+ = \tfrac{1}{2}a^{\dagger 2}, \qquad K_- = \tfrac{1}{2}a^2, \qquad K_0 = \tfrac{1}{2}a^\dagger a + \tfrac{1}{4}, \qquad K^2 = -\tfrac{3}{16}\mathbf{1}. \tag{213} $$

It is seen that $k(k-1) = -3/16$ which means that we have two irreducible representations from the continuous series with $k = 1/4$ and $k = 3/4$. Acting with K_0 on the number eigenstates of the harmonic oscillator, we easily see that the space $\mathcal{H}_{1/4}$ consists of the even harmonic oscillator number states, and the space $\mathcal{H}_{3/4}$ consists of the odd harmonic oscillator number states:

$$ \left|N;\tfrac{1}{4}\right\rangle_n \to |2N\rangle_n, \qquad \left|N;\tfrac{3}{4}\right\rangle_n \to |2N+1\rangle_n. \tag{214} $$

In other words, $\mathcal{H}_{1/4}$ is isomorphic to the space $\mathcal{H}_{\mathrm{even}}$ which we introduced earlier in the harmonic oscillator context, and $\mathcal{H}_{3/4}$ is isomorphic to the space $\mathcal{H}_{\mathrm{odd}}$ [143].

We next consider the squeezing operator

$$ T(r,\theta,\lambda) = \exp\left[-\frac{1}{4}r\,\mathrm{e}^{-\mathrm{i}\theta}a^{\dagger 2} + \frac{1}{4}r\,\mathrm{e}^{\mathrm{i}\theta}a^2\right]\exp\left[\mathrm{i}\frac{\lambda}{2}\left(a^\dagger a + \frac{1}{2}\right)\right], \tag{215} $$

which has been used extensively in quantum optics [144]. Substitution of equation (213) into the general $SU(1,1)$ operator of equation (171) shows that

$$ S\big(r,\theta,\lambda;\tfrac{1}{4}\big) = T(r,\theta,\lambda)\Pi_{\mathrm{even}}, \qquad S\big(r,\theta,\lambda;\tfrac{3}{4}\big) = T(r,\theta,\lambda)\Pi_{\mathrm{odd}}, \tag{216} $$

where Π_{even} and Π_{odd} are the projection operators into $\mathcal{H}_{\mathrm{even}}$ and $\mathcal{H}_{\mathrm{odd}}$, respectively, introduced earlier in equation (31). We note that both of these projection operators commute with $T(r,\theta,\lambda)$:

$$ [T(r,\theta,\lambda),\Pi_{\mathrm{even}}] = [T(r,\theta,\lambda),\Pi_{\mathrm{odd}}] = 0. \tag{217} $$

The $SU(1,1)$ coherent states introduced earlier become in the present context

$$ \begin{aligned} \left|z;\tfrac{1}{4}\right\rangle_{\mathrm{c}} &= T(r,\theta,\lambda)|0\rangle_n, \\ \left|z;\tfrac{3}{4}\right\rangle_{\mathrm{c}} &= T(r,\theta,\lambda)|1\rangle_n \end{aligned} \tag{218} $$

for the representations $k = 1/4$ and $k = 3/4$, respectively. The state $T(r,\theta,\lambda)|0\rangle_n$ is known as squeezed vacuum in quantum optics and is a superposition of even number states (it belongs to the space $\mathcal{H}_{\mathrm{even}}$). The state $T(r,\theta,\lambda)|1\rangle_n$ is a superposition of odd number states (it belongs to the space $\mathcal{H}_{\mathrm{odd}}$).

The Barut–Girardello states become in the present context

$$\left|\frac{z^2}{2};\frac{1}{4}\right\rangle_{\mathrm{BG}} = [2(1+\mathrm{e}^{-2|z|^2})]^{-1/2}(|z\rangle_{\mathrm{c}} + |-z\rangle_{\mathrm{c}}),$$
$$\left|\frac{z^2}{2};\frac{3}{4}\right\rangle_{\mathrm{BG}} = [2(1-\mathrm{e}^{-2|z|^2})]^{-1/2}(|z\rangle_{\mathrm{c}} - |-z\rangle_{\mathrm{c}}) \tag{219}$$

for the representations $k = 1/4$ and $k = 3/4$, respectively. They are superpositions of coherent states, which in quantum optics are known as even and odd coherent states [145].

We next introduce a generalization of the squeezing operator which we call 'parity-dependent squeezing operator'

$$\tilde{T}(r_0, \theta_0, \lambda_0; r_1, \theta_1, \lambda_1) = S\big(r_0, \theta_0, \lambda_0; \tfrac{1}{4}\big) + S\big(r_1, \theta_1, \lambda_1; \tfrac{3}{4}\big). \tag{220}$$

It acts with different parameters on the even and odd subspaces of the harmonic oscillator Hilbert space. In the special case that $r_0 = r_1$ and $\theta_0 = \theta_1$ and $\lambda_0 = \lambda_1$, $\tilde{T}$ reduces to the usual squeezing operator T.

The properties of the operators $\tilde{T}$ and the corresponding generalized squeezed states have been studied in [146]. The formalism can be used in the study of the time evolution of systems with the 'parity-dependent squeezed oscillator Hamiltonian':

$$H = \omega a^\dagger a + \Pi_{\mathrm{even}}(g_0 a^{\dagger 2} + g_0^* a^2) + \Pi_{\mathrm{odd}}(g_1 a^{\dagger 2} + g_1^* a^2). \tag{221}$$

In the special case $g_0 = g_1$, this Hamiltonian reduces to the 'squeezed oscillator Hamiltonian':

$$H = \omega a^\dagger a + g a^{\dagger 2} + g^* a^2. \tag{222}$$

8.2. *The $k = 1/2$ representation and harmonic oscillator phase states*

One realization of the $SU(1, 1)$ generators in terms of the harmonic oscillator creation and annihilation operators $a^\dagger$, a is

$$K_+ = n^{1/2} a^\dagger, \qquad K_- = a n^{1/2}, \qquad K_0 = n + \tfrac{1}{2}, \qquad K^2 = -\tfrac{1}{4}\mathbf{1}, \tag{223}$$

where $n = a^\dagger a$ is the number operator. In this case, $k(k-1) = -1/4$ and therefore this is the $k = 1/2$ representation. The correspondence between the $SU(1, 1)$ number states $|N; 1/2\rangle_n$ in the space $\mathcal{H}_{1/2}$ and the harmonic oscillator number states $|N\rangle_n$ is

$$|N; \tfrac{1}{2}\rangle_n \to |N\rangle_n. \tag{224}$$

A polar decomposition of the operators K_+, K_-, K_0 shows that in this special case $K_{\mathrm{r}} = n$ and that the $SU(1, 1)$ exponential of the phase operators E_+, E_- (defined through equation (169)) is the same as the harmonic oscillator exponential of the phase operators (defined through equation (20)). Consequently, the $SU(1, 1)$ phase states of equation (201) are also harmonic oscillator phase states, i.e., they are eigenstates of the harmonic oscillator exponential of the phase operators E_-:

$$|z\rangle_{\mathrm{ph}} = (1-|z|^2)^{1/2}\sum_{N=0}^{\infty} z^N |N\rangle_n = (1-|z|^2)^{1/2}(\mathbf{1} - zE_+)^{-1}|0\rangle_n, \qquad |z| < 1. \tag{225}$$

The bounded operator $(\mathbf{1} - zE_+)^{-1}$ has been defined in equation (202). The harmonic oscillator phase states behave in an isomorphic way to the $SU(1, 1)$ phase states and for this reason we use the same notation.

Phase states and phase operators have been studied for a long time using various approaches [147–157]. These techniques are not discussed here. The approach reviewed here is inspired by the theory of shift operators in functional analysis. It introduces the phase states of equation (225) in the unit disc, which will be used later to define analytic representations, which is the general theme of this review.

In the rest of this section, we define the angle-number phase space [60] which is similar but weaker concept in comparison to the x–p phase space. It has weaker properties and it is a less powerful mathematical tool. This is related to the fact that N does not take all integer values; it only takes the non-negative ones. We introduce displacement operators in the angle-number phase space as

$$\begin{aligned}
&W(N,\beta,\gamma) = E_+^N \exp(\mathrm{i}\beta n)\exp(\mathrm{i}\gamma),\\
&W(N_1,\beta_1,\gamma_1)W(N_2,\beta_2,\gamma_2) = W(N_1+N_2,\beta_1+\beta_2,\gamma_1+\gamma_2+N_2\beta_1),\\
&W(0,0,0)=\mathbf{1},\qquad W^\dagger W=\mathbf{1},\qquad WW^\dagger = \mathbf{1}-\sum_{M=0}^{N-1}|M\rangle_{n\,n}\langle M|,
\end{aligned}\tag{226}$$

where β and γ are real parameters and N is a non-negative integer. The operators W are isometric but not unitary. They do not have an inverse and therefore they form a semigroup. Examples of the action of these operators on number and phase states are given below:

$$\begin{aligned}
&E_+^N|M\rangle_n = |N+M\rangle_n, \qquad &&\exp(\mathrm{i}\beta n)|N\rangle = \exp(\mathrm{i}\beta N)|N\rangle_n,\\
&E_-^N|z\rangle_{\mathrm{ph}} = z^N|z\rangle_{\mathrm{ph}}, \qquad &&\exp(\mathrm{i}\beta n)|z\rangle_{\mathrm{ph}} = |z\,\mathrm{e}^{\mathrm{i}\beta}\rangle_{\mathrm{ph}}.
\end{aligned}\tag{227}$$

8.3. The Schwinger representation of SU(1, 1)

The Schwinger representation of $SU(1,1)$ involves two harmonic oscillators and is given by

$$K_+ = a_1^\dagger a_2^\dagger,\qquad K_- = a_1a_2,\qquad K_0 = \tfrac{1}{2}\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right).\tag{228}$$

We can easily check that they obey the $SU(1,1)$ commutation relations. Generalizations of this formalism to other groups have been discussed in [158] and they have many applications in various areas of physics (e.g., the interacting boson model in nuclear physics [159]).

If n_{d} is the difference of the number operators of the two oscillators, then the Casimir operator is given by

$$\begin{aligned}
&K^2 = \tfrac{1}{4}n_{\mathrm{d}}^2 - \tfrac{1}{4},\qquad n_{\mathrm{d}} = a_1^\dagger a_1 - a_2^\dagger a_2,\\
&[n_{\mathrm{d}},K_0] = [n_{\mathrm{d}},K_+] = [n_{\mathrm{d}},K_-] = 0.
\end{aligned}\tag{229}$$

Let $\mathcal{H}_1\times\mathcal{H}_2$ be the Hilbert space describing the two oscillators. We consider its subspaces

$$\mathcal{H}_m = \mathrm{span}\{|N+m,N\rangle_n;\, N\geqslant \max(0,-m)\},\tag{230}$$

where m is an integer. N takes the values $0,1,2,\ldots$ if m is positive, and the values $-m,-m+1,\ldots$ if m is negative. It is clear that the difference of the average number of photons in the two oscillators, for all states in $\mathcal{H}_m$, is equal to m.

We act with the operators of equations (228) and (229) on the states in the space $\mathcal{H}_m$ and we get

$$\begin{aligned}
&K_0|N+m,N\rangle_n = \left(N+\frac{m+1}{2}\right)|N+m,N\rangle_n,\\
&K^2|N+m,N\rangle_n = \frac{m^2-1}{4}|N+m,N\rangle_n.
\end{aligned}\tag{231}$$

Comparison with the general relations in equation (168) for the discrete series of the $SU(1,1)$ representations shows that the operators of equation (228) acting on the space $\mathcal{H}_m$ form the $k = (1+|m|)/2$ representation. It also shows that the harmonic oscillator number states $|N_1,N_2\rangle_n$ are also the $SU(1,1)$ number states $|N;k\rangle_n$:

$$|N_1,N_2\rangle_n = |N;k\rangle_n,\qquad N=\min(N_1,N_2),\qquad k=\tfrac{1}{2}(1+|N_1-N_2|).\tag{232}$$

The full Hilbert space $\mathcal{H}_1 \times \mathcal{H}_2$ is the direct sum of all the spaces $\mathcal{H}_m$ (it consists of one $k = 1/2$ representation, two $k = 1$ representations, two $k = 3/2$ representations, etc). We call Π_m the projection operators to the spaces $\mathcal{H}_m$. It is clear that

$$[\Pi_m, K_0] = [\Pi_m, K_+] = [\Pi_m, K_-] = [\Pi_m, n_\mathrm{d}] = 0. \tag{233}$$

Therefore, any operator which is a function of the operators K_0, K_-, K_+, n_d leaves the Hilbert spaces $\mathcal{H}_m$ invariant (i.e., acting on a state in $\mathcal{H}_m$ produces another state which belongs entirely in the same Hilbert space).

In many quantum optics problems related to amplifiers [160–162], we have the two-mode Hamiltonian

$$h = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \lambda a_1 a_2 + \lambda^* a_1^\dagger a_2^\dagger, \tag{234}$$

which can be written in terms of the $SU(1,1)$ generators of equation (228) as

$$h = (\omega_1 + \omega_2)\left(K_0 - \tfrac{1}{2}\right) + \tfrac{1}{2}(\omega_1 - \omega_2) n_\mathrm{d} + \lambda K_- + \lambda^* K_+. \tag{235}$$

This Hamiltonian leaves invariant the spaces $\mathcal{H}_m$. The time evolution of these systems can be studied using the analytic methods discussed below.

9. Analytic representations in the unit disc based on $SU(1,1)$ coherent states

9.1. States

We consider an arbitrary normalized state in $\mathcal{H}_k$:

$$|f\rangle = \sum_{N=0}^{\infty} f_N |N, k\rangle_n, \qquad \sum_{N=0}^{\infty} |f_N|^2 = 1. \tag{236}$$

This state is represented by the function [55–63]

$$f_\mathrm{H}(z; k) = \sum_{N=0}^{\infty} f_N d_\mathrm{H}(N; k) z^N = (1 - |z|^2)^{-k}{}_\mathrm{c}\langle z^*; k|f\rangle = (1 - |z|^2)^{-k} \langle f^*|z; k\rangle_\mathrm{c}, \tag{237}$$

which is analytic in the unit disc. The index 'H' stands for hyperbolic. The coefficients $d_\mathrm{H}(N; k)$ have been defined in equation (178).

Let ℓ_0 be an anticlockwise contour around the origin, within the unit disc. The fact that $f_\mathrm{H}(z; k)$ is analytic leads to the following relation:

$$\oint_{\ell_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{f_\mathrm{H}(z; k)}{d_\mathrm{H}(N; k) z^{N+1}} = f_N, \tag{238}$$

which gives the coefficients f_N. Relations analogous to equations (92) and (93) also hold here (the contours here need to be within the unit disc).

Using the resolution of the identity of equation (186), we find that the scalar product of two such states is given by

$$\langle f|g\rangle = \frac{2k - 1}{\pi} \int_D [f_\mathrm{H}(z; k)]^* g_\mathrm{H}(z; k)(1 - |z|^2)^{2k} \,\mathrm{d}\mu_\mathrm{H}(z). \tag{239}$$

As we already explained in equation (189), in the case $k = 1/2$ we take the one-sided limit of this formula. We recall here that for $k = 1/2$ the $SU(1,1)$ coherent states are the same as the $SU(1,1)$ phase states (equation (204)). We will show later that in the $k = 1/2$ case the present analytic formalism based on $SU(1,1)$ coherent states reduces to another analytic formalism based on phase states.

The Bergman space [54] consists of analytic functions $g(z)$ in the unit disc such that the following integral converges:

$$\frac{1+\alpha}{\pi}\int_D |g(z)|^p(1-|z|^2)^\alpha\,\mathrm{d}z_\mathrm{R}\,\mathrm{d}z_\mathrm{I} < \infty, \qquad -1<\alpha, \quad p>0. \tag{240}$$

Therefore, our functions $f_\mathrm{H}(z;k)$ belong to a Bergman space with $\alpha = 2k-2$ and $p=2$. Another way to express this is to define the growth of the function $g(z)$ near the unit circle as [54]

$$t = \lim_{|z|\to 1-}\sup\frac{\log|g(z)|}{\log\frac{1}{1-|z|}}. \tag{241}$$

Equation (239) shows that in our case $t \leqslant k-1$.

As examples, we easily show that for the $SU(1,1)$ number states

$$|N;k\rangle_n \to f_\mathrm{H}(z;k) = d_\mathrm{H}(N;k)z^N, \tag{242}$$

for the $SU(1,1)$ coherent states

$$|w;k\rangle_\mathrm{c} \to f_\mathrm{H}(z;k) = \frac{(1-|w|^2)^k}{(1-zw)^{2k}}, \tag{243}$$

for the $SU(1,1)$ phase states (in the space $\mathcal{H}_k$)

$$|w\rangle_\mathrm{ph} \to f_\mathrm{H}(z;k) = (1-|w|^2)^{1/2}\sum_{N=0}^{\infty} d_\mathrm{H}(N;k)(zw)^N \tag{244}$$

and for the Barut–Girardello states

$$|w;k\rangle_\mathrm{BG} \to f_\mathrm{H}(z;k) = \frac{w^{k-1/2}\exp(wz)}{[I_{2k-1}(2|w|)\Gamma(2k)]^{1/2}}. \tag{245}$$

The operators K_+, K_-, K_0 can be represented with the differential operators

$$K_+ = z^2\partial_z + 2kz, \qquad K_0 = z\partial_z + k, \qquad K_- = \partial_z. \tag{246}$$

We can easily check that these operators acting on $d_\mathrm{H}(N;k)z^N$ give the results of equation (168).

$SU(1,1)$ transformations in the unit disc are implemented through the Möbius conformal mappings

$$w(z) = \frac{az+b}{b^*z+a^*}, \qquad |a|^2-|b|^2 = 1. \tag{247}$$

Then the transformations

$$|f\rangle \to S(r,\theta,\lambda;k)|f\rangle \tag{248}$$

are implemented as

$$f_\mathrm{H}(z;k) \to f_\mathrm{H}(w;k)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^k = f_\mathrm{H}\left(\frac{az+b}{b^*z+a^*};k\right)(b^*z+a^*)^{-2k}, \tag{249}$$

where the parameters r,θ,λ are related to the parameters a,b through the relations

$$a = \mathrm{e}^{\mathrm{i}\lambda/2}\cosh\frac{r}{2}, \qquad b = \mathrm{e}^{\mathrm{i}\theta}\sinh\frac{r}{2}. \tag{250}$$

The 'multiplier' $(\mathrm{d}w/\mathrm{d}z)^k$ has been studied in detail by Bargmann [16].

The infinitesimal version of the mapping of equation (247) is

$$w = z + \epsilon_- + \epsilon_0 z + \epsilon_+ z^2, \tag{251}$$

where $\epsilon_-, \epsilon_0, \epsilon_+$ are infinitesimals. In this case

$$f_{\rm H}(z; k) \to [1 + \epsilon_- K_- + \epsilon_0 K_0 + \epsilon_+ K_+] f_{\rm H}(z; k), \tag{252}$$

where K_0, K_-, K_+ are the differential operators of equation (246). We note that the term $2kz$ in the operator K_+ is an infinitesimal multiplier. The same is true for the term k in the operator K_0.

We next comment on the zeros of these functions. The general theory of the density of zeros of functions in Bergman spaces is still an open problem in pure mathematics. The angular distribution of the zeros becomes important in this case. Recent results are summarized in [163, 164] and their possible significance for physics is an open problem. Here we make some general comments, analogous to those for the Bargmann functions in the Euclidean case.

Equation (237) shows that if ζ is a zero of the function $f_{\rm H}(z; k)$ then the $SU(1, 1)$ coherent state $|\zeta; k\rangle_{\rm c}$ is orthogonal to the state $|f^*\rangle$. A more general result is that if ζ is a zero of $f_{\rm H}(z; k)$ with multiplicity M, then $|f^*\rangle$ is orthogonal to the states $K_+^N|\zeta; k\rangle_{\rm c}$ with $N = 0, \ldots, M-1$. The proof is analogous to equation (125) for the Euclidean case. Another result which is similar to the Euclidean one is that if $\{\zeta_N\}$ is a sequence which has a limit w in the unit disc then the coherent states $\{|\zeta_N; k\rangle_{\rm c}\}$ form an overcomplete set.

9.2. Operators

An arbitrary operator $\mathcal{U}$ with matrix elements $\mathcal{U}_{MN} = {}_n\langle M; k|\mathcal{U}|N; k\rangle_n$ is represented by the kernel

$$\begin{aligned}\mathcal{K}_{\rm H}(z, \zeta^*; \mathcal{U}) &\equiv [(1 - |z|^2)(1 - |\zeta|^2)]^{-k}{}_{\rm c}\langle z^*; k|\mathcal{U}|\zeta^*; k\rangle_{\rm c} \\ &= \sum_{M,N=0}^{\infty} d_{\rm H}(M; k) d_{\rm H}(N; k)\mathcal{U}_{MN} z^M \zeta^{*N}\end{aligned} \tag{253}$$

and the state $|s\rangle = \mathcal{U}|f\rangle$ is represented by the function

$$s_{\rm H}(z; k) = \frac{2k-1}{\pi}\int_D \mathcal{K}_{\rm H}(z, \zeta^*; \mathcal{U}) f_{\rm H}(\zeta; k)(1 - |\zeta|^2)^{2k}\, {\rm d}\mu_{\rm H}(\zeta). \tag{254}$$

The kernel $\mathcal{K}_{\rm H}(z, \zeta^*; \mathcal{U})$ is an analytic function of z and ζ^* in the unit disc. Consequently, its diagonal component $\mathcal{K}_{\rm H}(z, z^*; \mathcal{U})$ determines uniquely through analytic continuation $\mathcal{K}_{\rm H}(z, \zeta^*; \mathcal{U})$.

As an example, we consider the unit operator $\mathbf{1}$ for which

$$\mathcal{K}_{\rm H}(z, \zeta^*; \mathbf{1}) = (1 - z\zeta^*)^{-2k}. \tag{255}$$

This is the reproducing kernel in the sense that when it acts on a state it gives the same state:

$$\frac{2k-1}{\pi}\int_D \mathcal{K}_{\rm H}(z, \zeta^*; \mathbf{1}) f_{\rm H}(\zeta; k)(1 - |\zeta|^2)^{2k}\, {\rm d}\mu_{\rm H}(\zeta) = f_{\rm H}(z; k). \tag{256}$$

We also consider the operators K_0, K_+, K_- which are represented by the kernels

$$\begin{aligned}\mathcal{K}_{\rm H}(z, \zeta^*; K_0) &= \frac{2k[-(z\zeta^*)^2 + z\zeta^* + 1]}{(1 - z\zeta^*)^{2k}}, \\ \mathcal{K}_{\rm H}(z, \zeta^*; K_+) &= \frac{2kz}{(1 - z\zeta^*)^{2k-1}}, \\ \mathcal{K}_{\rm H}(z, \zeta^*; K_-) &= \frac{2k\zeta^*}{(1 - z\zeta^*)^{2k-1}}.\end{aligned} \tag{257}$$

We can check that these integral representations are consistent with the differential representations of equation (246).

The trace of an operator $\mathcal{U}$ is given by

$$\mathrm{Tr}\,\mathcal{U} = \frac{2k-1}{\pi}\int_D \mathcal{K}_{\mathrm{H}}(z, z^*; \mathcal{U})(1-|z|^2)^{2k}\,\mathrm{d}\mu_{\mathrm{H}}(z). \tag{258}$$

The product of two operators $\mathcal{U}_1\mathcal{U}_2$ is represented with the kernel

$$\mathcal{K}_{\mathrm{H}}(z, \zeta^*; \mathcal{U}_1\mathcal{U}_2) = \frac{2k-1}{\pi}\int_D \mathcal{K}_{\mathrm{H}}(z, w^*; \mathcal{U}_1)\mathcal{K}_{\mathrm{H}}(w, \zeta^*; \mathcal{U}_2)(1-|w|^2)^{2k}\,\mathrm{d}\mu_{\mathrm{H}}(w). \tag{259}$$

We note that we can define slightly different kernels for the representation of the various operators. With a trivial revision of the above formulae, they all lead to the same final results. For example, below we will use the following kernel:

$$\mathcal{L}_{\mathrm{H}}(z, \zeta^*; \mathcal{U}) = \mathcal{K}_{\mathrm{H}}(z, \zeta^*; \mathcal{U})(1-z\zeta^*)^{2k}. \tag{260}$$

9.3. Hyperbolic Berezin formalism

In this section, we represent the various operators with the $\mathcal{L}$-kernels of equation (260). Using equation (259) we show that the diagonal part of the kernel representing the product $\mathcal{U}_1\mathcal{U}_2$ of two operators is given by

$$\mathcal{L}_{\mathrm{H}}(z, z^*; \mathcal{U}_1\mathcal{U}_2) = \frac{2k-1}{\pi}\int_D \mathcal{G}_{\mathrm{H}}(z, w; k)\mathcal{L}_{\mathrm{H}}(z, w^*; \mathcal{U}_1)\mathcal{L}_{\mathrm{H}}(w, z^*; \mathcal{U}_2)\,\mathrm{d}\mu_{\mathrm{H}}(w), \tag{261}$$

where $\mathcal{G}_{\mathrm{H}}(z, w; k)$ has been given in equation (184). Berezin [30] has proved that

$$\frac{2k-1}{\pi}\int_D \mathcal{G}_{\mathrm{H}}(z, w; k) f(w, w^*)\,\mathrm{d}\mu_{\mathrm{H}}(w) = \chi\left(\Delta_z^{(\mathrm{H})}\right) f(z, z^*), \tag{262}$$

where the hyperbolic Laplace–Beltrami operator $\Delta_z^{(\mathrm{H})}$ has been given in equation (9) and

$$\chi\left(\Delta_z^{(\mathrm{H})}\right) = \prod_{N=0}^{\infty}\left(1-\lambda_N\Delta_z^{(\mathrm{H})}\right)^{-1}, \qquad \lambda_N = \frac{1}{2k+N-1} - \frac{1}{2k+N}. \tag{263}$$

Equation (262) is the hyperbolic analogue of equation (85) in the Euclidean case. Therefore, equation (261) can be written as

$$\mathcal{L}_{\mathrm{H}}(z, z^*; \mathcal{U}_1\mathcal{U}_2) = \left[\chi\left(\Delta_\zeta^{(\mathrm{H})}\right)\mathcal{L}_{\mathrm{H}}(z, \zeta^*; \mathcal{U}_1)\mathcal{L}_{\mathrm{H}}(\zeta, z^*; \mathcal{U}_2)\right]_{\zeta=z}. \tag{264}$$

In the limit $k \to \infty$, the coefficients λ_N are close to zero, and we can write approximately

$$\chi\left(\Delta_z^{(\mathrm{H})}\right) \approx \prod_{N=0}^{\infty}\left(1+\lambda_N\Delta_z^{(\mathrm{H})}\right) = 1 + \frac{1}{2k-1}\Delta_z^{(\mathrm{H})} + \cdots. \tag{265}$$

Therefore,

$$\begin{aligned}\mathcal{L}_{\mathrm{H}}(z, z^*; \mathcal{U}_1\mathcal{U}_2) &= \mathcal{L}_{\mathrm{H}}(z, z^*; \mathcal{U}_1)\mathcal{L}_{\mathrm{H}}(z, z^*; \mathcal{U}_2)\\ &\quad + \frac{(1-|z|^2)}{2k-1}\frac{\partial\mathcal{L}_{\mathrm{H}}(z, z^*; \mathcal{U}_1)}{\partial z^*}\frac{\partial\mathcal{L}_{\mathrm{H}}(z, z^*; \mathcal{U}_2)}{\partial z} + \cdots.\end{aligned} \tag{266}$$

The parameter $1/k$ plays similar role to the Planck constant. In the semiclassical limit $k \to \infty$, only the first term survives and we get the classical result that the operators commute:

$$\mathcal{L}_{\mathrm{H}}(z, z^*; \mathcal{U}_1\mathcal{U}_2) = \mathcal{L}_{\mathrm{H}}(z, z^*; \mathcal{U}_1)\mathcal{L}_{\mathrm{H}}(z, z^*; \mathcal{U}_2). \tag{267}$$

We next keep the first two terms in the expansion of equation (266) and show that that the commutator $[\mathcal{U}_1, \mathcal{U}_2]$ is represented by the function

$$\mathcal{L}_{\mathrm{H}}(z, z^*; [\mathcal{U}_1, \mathcal{U}_2]) = -\frac{\mathrm{i}}{2k-1}\{\mathcal{L}_{\mathrm{H}}(z, z^*; \mathcal{U}_1), \mathcal{L}_{\mathrm{H}}(z, z^*; \mathcal{U}_2))\}_{\mathrm{H}}, \tag{268}$$

where the hyperbolic Poisson bracket has been given in equation (8). It is seen that when the higher order terms in the $1/k$ expansion are turned off, the quantum-mechanical commutator reduces to the hyperbolic Poisson bracket.

We note that similar formalism can be applied to equation (192) which connects the P and Q representations. In this case, we get

$$Q(z;\mathcal{U}) = P(z;\mathcal{U}) + \frac{1}{2k-1}\Delta_z^{(\mathrm{H})}P(z;\mathcal{U}) + \cdots. \tag{269}$$

As we have already mentioned in the Euclidean case, these semiclassical expressions assume that the relevant functions are smooth functions of $1/k$ so that the expansions of equations (266) and (269) are valid.

10. Analytic representations in the unit disc based on phase states and $\mathcal{Z}$-transform

10.1. States and operators

The formalism in this section is for any of the spaces $\mathcal{H}_k$ or for the harmonic oscillator Hilbert space $\mathcal{H}$. The results are isomorphic and for this reason we omit k in the notation. We consider an arbitrary normalized state

$$|f\rangle = \sum_{N=0}^{\infty} f_N|N\rangle_n, \qquad \sum_{N=0}^{\infty}|f_N|^2 = 1. \tag{270}$$

This state is represented by the function

$$f_{\mathcal{Z}}(z) = \sum_{N=0}^{\infty} f_N z^N = (1-|z|^2)^{-1/2}\,{}_{\mathrm{ph}}\langle z^*|f\rangle = (1-|z|^2)^{-1/2}\langle f^*|z\rangle_{\mathrm{ph}}, \tag{271}$$

which is analytic in the unit disc. This function with z replaced by $1/z$ (i.e., in the exterior of the unit disc) has been used extensively in digital signal processing [66] under the name $\mathcal{Z}$-transform, and for this reason we use the index '$\mathcal{Z}$' in the notation.

As examples, we easily show that for the number states

$$|N\rangle_n \rightarrow f_{\mathcal{Z}}(z) = z^N \tag{272}$$

and for the phase states

$$|w\rangle_{\mathrm{ph}} \rightarrow f_{\mathcal{Z}}(z) = \frac{(1-|w|^2)^{1/2}}{1-zw}. \tag{273}$$

In this representation, the exponential of the phase operators E_+, E_- and the operator $(\mathbf{1}-\zeta E_+)^{-1}$ given in equation (202) are represented as follows:

$$\begin{aligned} &E_- f_{\mathcal{Z}}(z) = \frac{f_{\mathcal{Z}}(z) - f_{\mathcal{Z}}(0)}{z}, \qquad E_+ f_{\mathcal{Z}}(z) = z f_{\mathcal{Z}}(z),\\ &(\mathbf{1}-\zeta E_+)^{-1} f_{\mathcal{Z}}(z) = \frac{f_{\mathcal{Z}}(z)}{1-\zeta z}, \end{aligned} \tag{274}$$

where $|z| < 1$ and $|\zeta| < 1$.

The fact that $\sum_N |f_N|^2 = 1$ implies that the following function exists:

$$f_{\mathcal{Z}}(\mathrm{e}^{\mathrm{i}\theta}) = \lim_{r\to 1} f_{\mathcal{Z}}(r\,\mathrm{e}^{\mathrm{i}\theta}) = \sum_{N=0}^{\infty} f_N\,\mathrm{e}^{\mathrm{i}N\theta}. \tag{275}$$

This is called 'boundary function' because it describes the function $f_{\mathcal{Z}}(z)$ on the unit circle $|z| = 1$. It is easily seen that

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \mathbb{P}(\theta) = 1, \qquad \mathbb{P}(\theta) \equiv |f_{\mathcal{Z}}(\mathrm{e}^{\mathrm{i}\theta})|^2. \tag{276}$$

The function $\mathbb{P}(\theta)$ describes the phase distribution of the quantum state $|f\rangle$.

Equation (275) appears like a Fourier transform, but we stress that N takes only the non-negative integer values. A direct consequence of this is that

$$\begin{aligned} &\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, \mathrm{e}^{-\mathrm{i}N\theta} f_{\mathcal{Z}}(\mathrm{e}^{\mathrm{i}\theta}) = f_N, \qquad N \geqslant 0, \\ &\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, \mathrm{e}^{-\mathrm{i}N\theta} f_{\mathcal{Z}}(\mathrm{e}^{\mathrm{i}\theta}) = 0, \qquad N < 0. \end{aligned} \tag{277}$$

The scalar product of two states is given in terms of their boundary functions as

$$\langle f|g\rangle = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} [f_{\mathcal{Z}}(\mathrm{e}^{\mathrm{i}\theta})]^* g_{\mathcal{Z}}(\mathrm{e}^{\mathrm{i}\theta}). \tag{278}$$

The Hardy space $H_p(D)$ [65] consists of analytic functions $g(z)$ in the unit disc such that the following integral converges:

$$\sup_{0<r<1} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} |g(r\mathrm{e}^{\mathrm{i}\theta})|^p < \infty. \tag{279}$$

Therefore, our functions $f_{\mathcal{Z}}(z)$ belong to the Hardy space with $p = 2$. We note that in Hardy spaces convergence of the one-dimensional integral of equation (279) is required, while in Bergman spaces convergence of the two-dimensional integral in the unit disc of equation (240) is required.

We next show that the formalism developed earlier in terms of the analytic functions $f_{\mathrm{H}}(z; k)$ in the limit $k \to 1/2$ becomes the formalism studied here in terms of $f_{\mathcal{Z}}(z)$ functions. It is of course trivial to see that in this limit $f_{\mathrm{H}}(z; k)$ becomes the function $f_{\mathcal{Z}}(z)$, but we also need to show that the scalar product of equation (239) becomes the scalar product of equation (278). Indeed, we can show that

$$\lim_{k \to \frac{1}{2}+\epsilon} \frac{2k-1}{\pi} \int_D |g(z)|^2 (1-|z|^2)^{2k}\, \mathrm{d}\mu_{\mathrm{H}}(z) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} |g(\mathrm{e}^{\mathrm{i}\theta})|^2. \tag{280}$$

10.2. Analytic properties

Hardy spaces have powerful properties which have been studied extensively in mathematics [65], and here we review the basic ones. We first define the Cauchy kernel

$$\mathcal{C}(r, \theta) = (1 - r\,\mathrm{e}^{\mathrm{i}\theta})^{-1}, \tag{281}$$

the Poisson kernel

$$\mathcal{P}(r, \theta) = \mathrm{Re}[2\mathcal{C}(r, \theta) - 1] = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta} \tag{282}$$

and the conjugate Poisson kernel $\mathcal{Q}(r, \theta)$

$$\mathcal{Q}(r, \theta) = \mathrm{Im}[2\mathcal{C}(r, \theta) - 1] = \frac{2r\sin\theta}{1 + r^2 - 2r\cos\theta}. \tag{283}$$

The boundary function $f_{\mathcal{Z}}(\theta)$ determines uniquely the analytic function $f_{\mathcal{Z}}(z)$ in the unit disc. Using the Cauchy formula for analytic functions, we prove

$$f_{\mathcal{Z}}(r\,\mathrm{e}^{\mathrm{i}\theta}) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \mathcal{C}(r, \theta - \phi) f_{\mathcal{Z}}(\mathrm{e}^{\mathrm{i}\phi}). \tag{284}$$

We denote as $f_{\mathcal{Z}}^{(\mathrm{R})}(r\,\mathrm{e}^{\mathrm{i}\theta})$ and $f_{\mathcal{Z}}^{(\mathrm{I})}(r\,\mathrm{e}^{\mathrm{i}\theta})$ the real and imaginary parts of $f_{\mathcal{Z}}(r\,\mathrm{e}^{\mathrm{i}\theta})$. We can express them in terms of $f_{\mathcal{Z}}^{(\mathrm{R})}(\mathrm{e}^{\mathrm{i}\theta})$ at the boundary as

$$\begin{aligned} f_{\mathcal{Z}}^{(\mathrm{R})}(r\,\mathrm{e}^{\mathrm{i}\theta}) &= \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \mathcal{P}(r, \theta - \phi) f_{\mathcal{Z}}^{(\mathrm{R})}(\mathrm{e}^{\mathrm{i}\phi}), \\ f_{\mathcal{Z}}^{(\mathrm{I})}(r\,\mathrm{e}^{\mathrm{i}\theta}) &= \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \mathcal{Q}(r, \theta - \phi) f_{\mathcal{Z}}^{(\mathrm{R})}(\mathrm{e}^{\mathrm{i}\phi}). \end{aligned} \tag{285}$$

10.3. Inner and outer states

Any function in the Hardy space $H_2(D)$ can be factorized into a product of an inner and an outer function [65]:

$$f_{\mathcal{Z}}(z) = f_{\mathrm{in}}(z) f_{\mathrm{out}}(z). \tag{286}$$

The outer part of the function $f_{\mathcal{Z}}(z)$ is defined as

$$f_{\mathrm{out}}(z) = \exp[\Phi(z)], \tag{287}$$

where

$$\begin{aligned} \Phi(r\,\mathrm{e}^{\mathrm{i}\theta}) &= \int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{2\pi} [2\mathcal{C}(r, \theta - \phi) - 1] \ln|f_{\mathcal{Z}}(\mathrm{e}^{\mathrm{i}\phi})| \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{2\pi} [2\mathcal{C}(r, \theta - \phi) - 1] \ln|\mathbb{P}(\phi)|. \end{aligned} \tag{288}$$

It is seen that the phase distribution function $\mathbb{P}(\phi)$ defines uniquely the outer part of $f_{\mathcal{Z}}(z)$. The inner part of the function $f_{\mathcal{Z}}(z)$ is defined as

$$f_{\mathrm{in}}(z) = f_{\mathcal{Z}}(z) \exp[-\Phi(z)]. \tag{289}$$

Both the inner and outer functions are analytic in the unit disc.

It is known [65] that in the interior of the unit disc ($|z| < 1$)

$$|f_{\mathrm{in}}(z)| \leqslant 1, \qquad |f_{\mathcal{Z}}(z)| \leqslant |f_{\mathrm{out}}(z)| \tag{290}$$

and on the unit circle ($|z| = 1$)

$$|f_{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\theta})| = 1, \qquad |f_{\mathcal{Z}}(\mathrm{e}^{\mathrm{i}\theta})| = |f_{\mathrm{out}}(\mathrm{e}^{\mathrm{i}\theta})| = [\mathbb{P}(\theta)]^{1/2}. \tag{291}$$

Therefore, all phase information of a quantum state is in its outer part.

A state $|f\rangle$ represented by a function $f_{\mathcal{Z}}(z) = f_{\mathrm{out}}(z)$ (with $f_{\mathrm{in}}(z) = 1$) is called an outer state. In order to check if a state is outer, we can calculate the outer part $f_{\mathrm{out}}(z)$ from equations (287) and (288) and compare it with the full function $f_{\mathcal{Z}}(z)$. A simpler criterion is that a function is an outer function if and only if

$$\ln|f_{\mathrm{out}}(0)| = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \ln|f_{\mathrm{out}}(\mathrm{e}^{\mathrm{i}\theta})| = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \ln[\mathbb{P}(\theta)]. \tag{292}$$

We next apply this mathematical formalism into a physical context [60]. The phase states $|r\,\mathrm{e}^{\mathrm{i}\phi}\rangle_{\mathrm{ph}}$ have phase distribution function

$$\mathbb{P}(\theta) = \mathcal{P}(r, \theta + \phi), \tag{293}$$

where $\mathcal{P}$ is the Poisson kernel of equation (282). We note that

$$\lim_{r\to 1} \mathcal{P}(r, \theta) = 2\pi\delta(\theta) \tag{294}$$

and this provides an extra justification for the use of the term phase states for these states. Phase states have phase distribution which is a delta function near the unit circle, and which is smoothed out as we move towards the centre of the circle. Inserting equation (293) into equation (292), we confirm that the phase states are outer states.

A state $|f\rangle$ represented by a function $f_{\mathcal{Z}}(z) = f_{\text{in}}(z)$ (with $f_{\text{out}}(z) = 1$) is called an inner state. All inner states have uniform phase distribution $\mathbb{P}(\theta) = 1$. An example of an inner state is the number state $|N\rangle_n$ with $f_{\mathcal{Z}}(z) = z^N$. This is easily understood physically, because number states have uniform phase distribution. Another important inner function is

$$f_{\mathcal{Z}}(z) = \frac{\zeta - z}{1 - \zeta^* z}, \qquad |\zeta| < 1, \tag{295}$$

and it is easily seen that it represents the state

$$\begin{aligned} &\hat{\mathcal{B}}(\zeta)|0\rangle_n = [\zeta - E_+]|\zeta^*\rangle_{\text{ph}}, \\ &\hat{\mathcal{B}}(\zeta) = [\zeta - E_+](\mathbf{1} - \zeta^* E_+)^{-1}, \qquad |\zeta| < 1. \end{aligned} \tag{296}$$

$\hat{\mathcal{B}}(\zeta)$ is a bounded operator.

Equations (290) and (291) show that the inner functions map the unit disc in the unit disc ($|f_{\text{in}}(z)| \leqslant 1$ for $|z| < 1$) and the unit circle in the unit circle ($|f_{\text{in}}(\mathrm{e}^{\mathrm{i}\theta})| = 1$). It is easily proved that the most general bilinear transformations which do this are given in equation (295), and therefore they are the 'building blocks' of more general inner functions. Indeed, a product of a finite number of these functions is also an inner function and it represents the following state:

$$z^{p_0} \prod_{N=1}^{K} \left(\frac{\zeta_N - z}{1 - \zeta_N^* z}\right)^{p_N} \to (E_+)^{p_0} \prod_{N=1}^{K} [\hat{\mathcal{B}}(\zeta_N)]^{p_N} |0\rangle_n. \tag{297}$$

Here p_n are non-negative integers and ζ_n are distinct numbers in the unit disc. Below we will discuss infinite such products and the most general form of inner functions.

10.4. Zeros of the functions $f_{\mathcal{Z}}(z)$

Equation (271) shows that if ζ is a zero of the function $f_{\mathcal{Z}}(z)$ in the unit disc, then the phase state $|\zeta\rangle_{\text{ph}}$ is orthogonal to the corresponding state $|f^*\rangle$. All zeros of $f_{\mathcal{Z}}(z)$ are in its inner part; the outer part is an exponential and has no zeros.

It is known [65] that the zeros ζ_N of a bounded analytic function in the unit disc satisfy the conditions

$$\prod_{N=1}^{\infty} |\zeta_N|^{p_N} < \infty \leftrightarrow \sum_{N=1}^{\infty} p_N(1 - |\zeta_N|) < \infty. \tag{298}$$

Here p_N is the multiplicity of the zero ζ_N. This shows that the zeros move quickly towards the unit circle $|z| = 1$. An immediate consequence of this is that if a sequence ζ_N violates the conditions of equation (298), then the corresponding phase states $\{|\zeta_N\rangle_{\text{ph}}\}$ form an overcomplete set.

We next consider a sequence of complex numbers in the unit disc ζ_N and the corresponding multiplicities p_N, such that the condition of equation (298) is satisfied. We will construct inner functions with those zeros, and this will prove that the corresponding phase states form an undercomplete set. One inner function which has those zeros is the Blaschke product [65]

$$B(z) = z^{p_0} \prod_{N=1}^{\infty} \left(\frac{\zeta_N^*}{|\zeta_N|} \frac{\zeta_N - z}{1 - \zeta_N^* z} \right)^{p_N}. \tag{299}$$

This is similar to equation (297), but here we have an infinite product, and in addition to that we have the phase factors $\zeta_N^*/|\zeta_N|$ which are important for the convergence. The state represented by this function is

$$|s\rangle \equiv (E_+)^{p_0} \prod_{N=1}^{\infty} \left[\frac{\zeta_N^*}{|\zeta_N|} \hat{\mathcal{B}}(\zeta_N) \right]^{p_N} |0\rangle_n. \tag{300}$$

The most general inner function with the same zeros is of the form $B(z)\exp[h(z)]$ where $\exp[h(z)]$ is the so-called singular function and has no zeros. The properties of the singular functions are discussed in the literature [65]. It is therefore clear that the general function $f_{\mathcal{Z}}(z)$ can be written as

$$f_{\mathcal{Z}}(z) = f_{\text{in}}(z) f_{\text{out}}(z) = B(z)\exp[h(z)] f_{\text{out}}(z), \tag{301}$$

where only the Blaschke product $B(z)$ has zeros. We note that if the function $f_{\mathcal{Z}}(z)$ has a zero ζ_N, then it also has a pole $1/\zeta_N^*$ in the exterior of the unit disc. The converse is not true. If the function $f_{\mathcal{Z}}(z)$ has a pole at a point $1/\zeta_N^*$ in the exterior of the unit disc, it does not necessarily have a zero at ζ_N (see equation (273)).

A quantum state evolves in time, and the corresponding zeros also evolve in time. In order to give an example of this, we consider the realization of the $k = 1/2$ representation of $SU(1,1)$ given in equation (223), and the Hamiltonian

$$\begin{aligned} H &= -\frac{1}{2\mathrm{i}} r\,\mathrm{e}^{-\mathrm{i}\theta} K_+ + \frac{1}{2\mathrm{i}} r\,\mathrm{e}^{\mathrm{i}\theta} K_- = -\frac{1}{2\mathrm{i}} r\,\mathrm{e}^{-\mathrm{i}\theta} n E_+ + \frac{1}{2\mathrm{i}} r\,\mathrm{e}^{\mathrm{i}\theta} E_- n \\ &= -\frac{1}{2\mathrm{i}} r\,\mathrm{e}^{-\mathrm{i}\theta} n^{1/2} a^\dagger + \frac{1}{2\mathrm{i}} r\,\mathrm{e}^{\mathrm{i}\theta} a n^{1/2} \end{aligned} \tag{302}$$

acting on the states in the harmonic oscillator Hilbert space. We assume that at $t = 0$ the system described by this Hamiltonian is in the state $|f\rangle$, with the corresponding function $f_{\mathcal{Z}}(z)$ parametrized as in equations (301) and (299). Then at time t the state becomes $\exp[\mathrm{i}Ht]|f\rangle$ and according to equation (249), which should be used here with $k = 1/2$, is represented by the function

$$\begin{aligned} &\exp[\mathrm{i}Ht]|f\rangle \to f_{\mathcal{Z}}\left(\frac{a(t)z + b(t)}{b^*(t)z + a^*(t)} \right) (b^*(t)z + a^*(t))^{-1}, \\ &a(t) = \cosh\frac{rt}{2}, \qquad b(t) = \mathrm{e}^{\mathrm{i}\theta}\sinh\frac{rt}{2}. \end{aligned} \tag{303}$$

Therefore, in this example, the zeros evolve in time as

$$\zeta_N(t) = \frac{a^*(t)\zeta_N - b(t)}{-b^*(t)\zeta_N + a(t)}. \tag{304}$$

The study of the motion of the zeros can provide a deeper insight about the system. We stress, however, that knowledge of the zeros does not define uniquely the system. We have seen that the outer and singular parts of $f_{\mathcal{Z}}(z)$ have no zeros.

11. The Barut–Girardello analytic representation in the complex plane

11.1. States

Using the Barut–Girardello states we can represent the arbitrary state $|f\rangle$ of equation (236) which belongs to the Hilbert space $\mathcal{H}_k$ with the following function which is analytic in the complex plane [52, 53]:

$$f_{\mathrm{BG}}(z;k) = \sum_{N=0}^{\infty} d_{\mathrm{BG}}(N;k) f_N z^N = [I_{2k-1}(2|z|)]^{1/2} z^{1/2-k} \langle f^*|z;k\rangle_{\mathrm{BG}}. \tag{305}$$

The coefficients $d_{\mathrm{BG}}(N;k)$ have been defined in equation (209).

Using the resolution of the identity of equation (211), we easily see that the scalar product is given by

$$\langle f|g\rangle = \int_C \mathrm{d}\mu_{\mathrm{BG}}(z,k)|z|^{2k-1}[f_{\mathrm{BG}}(z;k)]^* g_{\mathrm{BG}}(z;k). \tag{306}$$

$\mathrm{d}\mu_{\mathrm{BG}}(z,k)$ contains the modified Bessel function $K_{2k-1}(2|z|)$ which at infinity behaves as $\exp(-2|z|)$. Convergence of the above integral shows that the growth of the $f_{\mathrm{BG}}(z;k)$ is smaller than $(\rho = 1, \sigma = 1)$.

We next give briefly results about the completeness of sequences of Barut–Girardello states using the same argument as with the Euclidean coherent states. If ζ is a zero of $f_{\mathrm{BG}}(z;k)$, then the Barut–Girardello state $|\zeta;k\rangle_{\mathrm{BG}}$ is orthogonal to the state $|f^*\rangle$. If $\{\zeta_N\}$ is a sequence of complex numbers which has a limit w, then the corresponding set of Barut–Girardello states $\{|\zeta_N;k\rangle_{\mathrm{BG}}\}$ is overcomplete.

We next consider a sequence $\{\zeta_N\}$ of the type given in equation (127). If the density (η,δ) of this sequence is greater than $(1,1)$, then the corresponding set of Barut–Girardello states $\{|\zeta_N;k\rangle_{\mathrm{BG}}\}$ is overcomplete, and if it is less than $(1,1)$ then it is undercomplete. As an example, we consider the one-dimensional lattice of Barut–Girardello states $\{|N\gamma;k\rangle_{\mathrm{BG}}\}$, where N takes all integer values and γ is a complex number. The density of the corresponding complex numbers is $\eta = 1$ and $\delta = 1/|\gamma|$. Therefore, these states form an overcomplete set when $|\gamma| < 1$, and an undercomplete set when $|\gamma| > 1$.

11.2. Operators

An arbitrary operator $\mathcal{U}$ with matrix elements $\mathcal{U}_{MN} = {}_n\langle M;k|\mathcal{U}|N;k\rangle_n$ is represented by the following kernel:

$$\begin{aligned}\mathcal{K}_{\mathrm{BG}}(z_1,z_2^*;\mathcal{U}) &\equiv [I_{2k-1}(2|z_1|)I_{2k-1}(2|z_2|)]^{1/2}(z_1z_2^*)^{1/2-k}\ {}_{\mathrm{BG}}\langle z_1^*;k|\mathcal{U}|z_2^*;k\rangle_{\mathrm{BG}} \\ &= \sum_{M,N=0}^{\infty} d_{\mathrm{BG}}(M;k)d_{\mathrm{BG}}(N;k)\mathcal{U}_{MN} z_1^M z_2^{*N}\end{aligned} \tag{307}$$

and the state $|s\rangle = \mathcal{U}|f\rangle$ is represented by the function

$$s_{\mathrm{BG}}(z) = \int_C \mathcal{K}_{\mathrm{BG}}(z,\zeta^*;\mathcal{U}) f_{\mathrm{BG}}(\zeta;k)|\zeta|^{2k-1}\,\mathrm{d}\mu_{\mathrm{BG}}(\zeta). \tag{308}$$

As an example, we consider the unit operator $\mathbf{1}$ for which

$$\mathcal{K}_{\mathrm{BG}}(z_1,z_2^*;\mathbf{1}) = \frac{{}_0F_1(2k;z_1z_2^*)}{\Gamma(2k)}, \tag{309}$$

where ${}_0F_1(2k;z_1z_2^*)$ is a generalized hypergeometric series. This is the reproducing kernel in the sense that when it acts on a state it gives the same state:

$$\int_C \mathcal{K}_{\mathrm{BG}}(z,\zeta^*;\mathbf{1}) f_{\mathrm{BG}}(\zeta;k)|\zeta|^{2k-1}\,\mathrm{d}\mu_{\mathrm{BG}}(\zeta) = f_{\mathrm{BG}}(z;k). \tag{310}$$

Another example is the operator K_0 for which we find

$$\mathcal{K}_{\mathrm{BG}}(z_1, z_2^*; K_0) = \frac{{}_0F_1(2k+1; z_1 z_2^*)}{\Gamma(2k+1)}(z_1 z_2^*) + \frac{{}_0F_1(2k; z_1 z_2^*)}{\Gamma(2k)}k. \tag{311}$$

The other $SU(1,1)$ generators can also be represented with kernels but they are more easily represented with the differential operators:

$$K_+ = z, \qquad K_- = 2k\partial_z + z\partial_z^2, \qquad K_0 = z\partial_z + k. \tag{312}$$

We can check that the differential representation is consistent with the integral representation with kernels.

11.3. Its relation to the analytic representation in the unit disc based on $SU(1,1)$ coherent states

In this section, we give transforms which connect the Barut–Girardello analytic representation in the complex plane with the analytic representation in the unit disc based on $SU(1,1)$ coherent states [60]. We consider a state $|f\rangle$ in $\mathcal{H}_k$ which is represented by the function $f_{\mathrm{H}}(z;k)$ defined in equation (237) and also by the function $f_{\mathrm{BG}}(\zeta;k)$ defined in equation (305). We can show that

$$f_{\mathrm{H}}(z,k) = \frac{1}{z^{2k}[\Gamma(2k)]^{1/2}} \int_0^\infty w^{2k-1} f_{\mathrm{BG}}(w;k)\,\mathrm{e}^{-w/z}\,\mathrm{d}w. \tag{313}$$

This is a Laplace transform which involves integration of the function $f_{\mathrm{BG}}(\zeta;k)$ along the positive real axis. The integral converges only when z is in the right half of the unit disc. Therefore, equation (313) gives the function $f_{\mathrm{H}}(z;k)$ in the right half of the unit disc, and through analytic continuation we can define it in the whole unit disc. The inverse transform is given by

$$f_{\mathrm{BG}}(w;k) = \frac{[\Gamma(2k)]^{1/2}}{w^{2k-1}} \frac{1}{2\pi\mathrm{i}} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} z^{-2k} f_{\mathrm{H}}\left(\frac{1}{z};k\right) \mathrm{e}^{wz}\,\mathrm{d}z. \tag{314}$$

The integration is along the line $z = 1 + \mathrm{i}z_{\mathrm{I}}$ where z_{I} is a real number. We note that when z takes values along this line $1/z$ is in the unit disc.

As an example, we consider the eigenvalue equation

$$(\alpha K_+ + \beta K_0 + \gamma K_-)|\lambda,k\rangle = \lambda|\lambda,k\rangle. \tag{315}$$

Here $|\lambda,k\rangle$ is an eigenvector of the operator $\alpha K_+ + \beta K_0 + \gamma K_-$ which in the analytic representation in the unit disc is represented with the function $f_{\mathrm{H}}(z,\lambda;k)$ and in the Barut–Girardello analytic representation in the complex plane is represented with the function $f_{\mathrm{BG}}(z,\lambda;k)$. Taking into account equations (246) and (312), we see that the eigenvalue equation (315) becomes the differential equations

$$[(\alpha z^2 + \beta z + \gamma)\partial_z + (2k\alpha z + \beta k - \lambda)] f_{\mathrm{H}}(z,\lambda;k) = 0, \tag{316}$$

$$\left[\gamma z\partial_z^2 + (\beta z + 2k\gamma)\partial_z + (\alpha z + \beta k - \lambda)\right] f_{\mathrm{BG}}(z,\lambda;k) = 0 \tag{317}$$

in the analytic representation in the unit disc and in the Barut–Girardello analytic representation in the complex plane, respectively. We can confirm that the transformations of equations (313) and (314) transform these equations to each other. The solutions of any of these two differential equations provide the eigenvectors $|\lambda,k\rangle$ in the corresponding representation. We will not present these solutions here [60, 165].

11.4. The Barut–Girardello analytic representations with $k = 1/4$ and $k = 3/4$ and their relation to the Bargmann representation

We consider the realization of the $SU(1,1)$ algebra in terms of harmonic oscillator creation and annihilation operators given in equation (213). We have explained that we have two irreducible representations of $SU(1,1)$ with $k = 1/4$ and $k = 3/4$. For $k = 1/4$ the Hilbert space $\mathcal{H}_{1/4}$ is isomorphic to the space $\mathcal{H}_{\rm even}$ that is spanned by the even number states of the harmonic oscillator, and for $k = 3/4$ the Hilbert space $\mathcal{H}_{3/4}$ is isomorphic to the space $\mathcal{H}_{\rm odd}$ that is spanned by the odd number states of the harmonic oscillator. An arbitrary state $|f\rangle$ in the harmonic oscillator Hilbert space $\mathcal{H}$ can be written as

$$|f\rangle = \Pi_{\rm even}|f\rangle + \Pi_{\rm odd}|f\rangle = \mathcal{N}_{\rm even}^{-1}|f_{\rm even}\rangle + \mathcal{N}_{\rm odd}^{-1}|f_{\rm odd}\rangle, \tag{318}$$

where $\mathcal{N}_{\rm even}$ and $\mathcal{N}_{\rm odd}$ are normalization factors so that the states $|f_{\rm even}\rangle$ and $|f_{\rm odd}\rangle$ are normalized to 1:

$$\mathcal{N}_{\rm even} = \langle f|\Pi_{\rm even}|f\rangle^{1/2}, \qquad \mathcal{N}_{\rm odd} = \langle f|\Pi_{\rm odd}|f\rangle^{1/2}. \tag{319}$$

The Barut–Girardello states in this case are the superpositions of two coherent states given in equation (219). Using this we can find a relation between the Bargmann representation $f_{\rm B}(z)$ of a harmonic oscillator state $|f\rangle$ and the Barut–Girardello representations for the states $|f_{\rm even}\rangle$ and $|f_{\rm odd}\rangle$ which we denote as $f_{\rm BG}(z, 1/4)$ and $f_{\rm BG}(z, 3/4)$, respectively [53, 60]:

$$f_{\rm B}(z) = \pi^{1/4}\mathcal{N}_{\rm even} f_{\rm BG}\left(\frac{z^2}{2}; \frac{1}{4}\right) + 2^{-1/2}\pi^{1/4}\mathcal{N}_{\rm odd} z f_{\rm BG}\left(\frac{z^2}{2}; \frac{3}{4}\right). \tag{320}$$

We have seen in equation (99) that in the Bargmann representation the creation and annihilation operators are z and ∂_z, respectively. Substitution of this into equation (213) gives the operators

$$K_+ = \tfrac{1}{2}z^2, \qquad K_- = \tfrac{1}{2}\partial_z^2, \qquad K_0 = \tfrac{1}{2}z\partial_z + \tfrac{1}{4}. \tag{321}$$

Taking into account that the Barut–Girardello functions in equation (320) contain $z^2/2$, we see that the operators in equation (321) are equivalent to the operators in equation (312), for $k = 1/4$. In order to understand this equivalence in the case $k = 3/4$, we also need to take into account that the corresponding Barut–Girardello function in equation (320) is multiplied by z.

12. Hyperbolic contour representation in the unit disc

12.1. Quantum states

In the hyperbolic contour formalism [50], the arbitrary ket state of equation (236) and the corresponding bra state $\langle f|$ are represented as

$$\begin{aligned} |f\rangle &\to f_{\rm Hk}(z;k) = \sum_N f_N d_{\rm H}(N;k) z^N, \\ \langle f| &\to f_{\rm Hb}(z;k) = \sum_n f_N^* d_{\rm H}(N;k)^{-1} z^{-N-1}, \end{aligned} \tag{322}$$

where the indices 'Hk' and 'Hb' refer to 'hyperbolic ket' and 'hyperbolic bra', respectively. The ket function $f_{\rm Hk}(z;k)$ is the same with the function $f_{\rm H}(z;k)$, equation (237), and is analytic in the unit disc.

In order to study the convergence of these two series, we calculate the absolute value of the ratio of two successive terms, for both of them:

$$r_N^{(\rm k)} = \frac{|f_{N+1}|s_N}{|f_N|}|z|, \qquad r_N^{(\rm b)} = \frac{|f_{N+1}|}{|f_N|s_N}\frac{1}{|z|}, \qquad s_N = \left[\frac{N+2k}{N+1}\right]^{1/2}. \tag{323}$$

The labels 'k' and 'b' indicate ket and bra, respectively. A series converges or diverges when the limit of this ratio is less or greater than 1, respectively. The limit of the ratio $|f_{N+1}|/|f_N|$ does not exceed 1, because $\sum |f_N|^2 = 1$. The limit of s_N is 1. Therefore, $f_{\mathrm{Ek}}(z;k)$ converges at least in the unit disc, and $f_{\mathrm{Eb}}(z;k)$ converges at least in the exterior of the unit disc $|z| > 1$. Our aim is to find an annulus in the neighbourhood of the unit circle $|z| = 1$ where both the ket function $f_{\mathrm{Hk}}(z;k)$ and the bra function $f_{\mathrm{Hb}}(z;k)$ converge, so that we can use them in contour integrals. In order to do this, we consider states in a subspace of the full Hilbert space $\mathcal{H}_k$ defined as

$$\mathcal{H}_k(\epsilon) = \left\{ |f\rangle : \lim_{N\to\infty} \frac{|f_{N+1}|}{|f_N|} \leqslant 1 - \epsilon \right\}. \tag{324}$$

When $\epsilon' > \epsilon$, then $\mathcal{H}_k(\epsilon')$ is a subspace of $\mathcal{H}_k(\epsilon)$. For states in $\mathcal{H}_k(\epsilon)$, the series of $f_{\mathrm{Hk}}(z;k)$ converges in the region $|z| < 1/(1-\epsilon)$, and the series of $f_{\mathrm{Hk}}(z;k)$ converges in the region $|z| > 1-\epsilon$. Therefore, both $f_k(z;k)$ and $f_b(z;k)$ converge at least within the annulus

$$\mathcal{R}(\epsilon) = \left\{ 1 - \epsilon < |z| < \frac{1}{1-\epsilon} \right\}. \tag{325}$$

The spaces $\mathcal{H}_\epsilon$ are *not* dense in the full Hilbert space (for any positive ϵ). There are many states for which the limit of the ratio $|f_{N+1}|/|f_N|$ is 1. Below we consider states in $\mathcal{H}_k(\epsilon)$ and use integrals along an anticlockwise contour ℓ which is in the annulus $\mathcal{R}(\epsilon)$.

The scalar product of two states $|f\rangle, |g\rangle$ in the space $\mathcal{H}_k(\epsilon)$ is given by

$$\langle f|g\rangle = \oint_{\ell\in\mathcal{R}(\epsilon)} \frac{\mathrm{d}z}{2\pi\mathrm{i}} f_{\mathrm{Hb}}(z;k) g_{\mathrm{Hk}}(z;k). \tag{326}$$

The ket function $f_{\mathrm{Hk}}(z;k)$ is related to the bra function $f_{\mathrm{Hb}}(z;k)$ through the formulae

$$\begin{aligned} &\oint_{\ell\in\mathcal{R}(\epsilon)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} f_{\mathrm{Hb}}(w;k)(1-z^*w)^{-2k} = [f_{\mathrm{Hk}}(z;k)]^*, \\ &\frac{2k-1}{z}\int_0^\infty \frac{\mathrm{d}t}{(1+t)^{n+2k}} \left[f_{\mathrm{Hk}}\left(\frac{t}{z^*};k\right)\right]^* = f_{\mathrm{Hb}}(z;k), \qquad k > \frac{1}{2}. \end{aligned} \tag{327}$$

As an example, we consider the $SU(1,1)$ coherent states $|w,k\rangle_{\mathrm{c}}$ and show that

$$\begin{aligned} f_{\mathrm{Hk}}(z;k) &= \frac{(1-|w|^2)^k}{(1-zw)^{2k}}, \qquad |z| < |w|^{-1}, \\ f_{\mathrm{Hb}}(z;k) &= (1-|w|^2)^k \sum_{n=0}^{\infty} \frac{w^n}{z^{n+1}} = \frac{(1-|w|^2)^k}{z-w}, \qquad |z| > |w|. \end{aligned} \tag{328}$$

12.2. Operators

An arbitrary operator $\mathcal{U}$ with matrix elements $\mathcal{U}_{MN} = {}_n\langle M|\mathcal{U}|N\rangle_n$ is represented by the following kernel:

$$\mathcal{C}_{\mathrm{H}}(z_1, z_2; \mathcal{U}) = \sum_{M,N} \frac{\mathcal{U}_{MN} d_{\mathrm{H}}(M,k) z_1^M}{d_{\mathrm{H}}(N,k) z_2^{N+1}}. \tag{329}$$

This function is useful if the double series converges for $z_1 \in \mathcal{S}$ and $z_2 \in \mathcal{T}$, where $\mathcal{S}$ and $\mathcal{T}$ are regions defined in equation (1) with radii that depend on the operator.

As examples, we give the following:

$$
\begin{aligned}
\mathcal{C}_{\mathrm{H}}(z_1, z_2; \mathbf{1}) &= \frac{1}{z_2 - z_1}, && |z_1| < |z_2|, \\
\mathcal{C}_{\mathrm{H}}(z_1, z_2; K_0) &= \frac{kz_2 + (1-k)z_1}{(z_2 - z_1)^2}, && |z_1| < |z_2|, \\
\mathcal{C}_{\mathrm{H}}(z_1, z_2; K_+) &= \frac{(1-2k)z_1^2 + 2kz_2 z_1}{(z_2 - z_1)^2}, && |z_1| < |z_2|, \\
\mathcal{C}_{\mathrm{H}}(z_1, z_2; K_-) &= \frac{1}{(z_2 - z_1)^2}, && |z_1| < |z_2|.
\end{aligned}
\tag{330}
$$

In all these examples, $z_1 \in \mathcal{S}(r)$ and $z_2 \in \mathcal{T}(r)$ for arbitrary r.

For a state $|f\rangle$ in $\mathcal{H}(\epsilon)$, the ket state $|g\rangle = \mathcal{U}|f\rangle$ and the bra state $\langle g| = \langle f|\mathcal{U}^\dagger$ are represented by

$$
\begin{aligned}
g_{\mathrm{Hk}}(z_1) &= \oint_{\ell \in \mathcal{T}(r)\cap\mathcal{R}(\epsilon)} \frac{\mathrm{d}z_2}{2\pi \mathrm{i}} \mathcal{C}_{\mathrm{H}}(z_1, z_2; \mathcal{U}) f_{\mathrm{Hk}}(z_2; k), \\
g_{\mathrm{Hb}}(z_2; k) &= \oint_{\ell \in \mathcal{R}(\epsilon)\cap\mathcal{S}(r)} \frac{\mathrm{d}z_1}{2\pi \mathrm{i}} f_{\mathrm{Hb}}(z_1; k) \mathcal{C}_{\mathrm{H}}(z_1, z_2; \mathcal{U}^\dagger).
\end{aligned}
\tag{331}
$$

In the first integral, $z_1 \in \mathcal{S}(r)$ and $z_2 \in \mathcal{T}(r) \cap \mathcal{R}(\epsilon)$ (and we have to choose $r < (1-\epsilon)^{-1}$). Of course, $g_{\mathrm{Hk}}(z_1)$ is analytic in the unit disc, and through analytic continuation it is defined in the whole unit disc. In the second integral, $z_1 \in \mathcal{R}(\epsilon) \cap \mathcal{S}(r)$ (in this case we have to choose $r > 1 - \epsilon$) and $z_2 \in \mathcal{T}(r)$.

We use equations (331) with any of the kernels in equation (330). The first integral involves integration over z_2 and the singularity of $\mathcal{C}_{\mathrm{E}}(z_1, z_2; \mathcal{U})$ at $z_2 = z_1$ is inside the contour ℓ. In contrast, the second integral involves integration over z_1 and the singularity of $\mathcal{C}_{\mathrm{E}}(z_1, z_2; \mathcal{U})$ at $z_1 = z_2$ is outside the contour ℓ; it is only the singularities of $f_{\mathrm{Hb}}(z_1)$ that contribute to this integral.

We have explained earlier that if $\{\zeta_N\}$ is a sequence which has a limit w in the unit disc, then the $SU(1,1)$ coherent states $\{|\zeta_N; k\rangle_{\mathrm{c}}\}$ form an overcomplete set. Therefore, the coherent states in a contour ℓ form a highly overcomplete set of states. In practice, it is not easy to find an expansion of an arbitrary state in terms of these coherent states, and the hyperbolic contour formalism provides one. An arbitrary state $|f\rangle \in \mathcal{H}_k(\epsilon)$ can be written as

$$
\begin{aligned}
|f\rangle &= \oint_{\ell \in \mathcal{R}(\epsilon)} \frac{\mathrm{d}z}{2\pi \mathrm{i}} (1 - |z|^2)^{-k} a(z) |z; k\rangle_{\mathrm{c}}, \\
a(z) &= \sum_N \frac{f_N}{d_{\mathrm{H}}(N; k) z^{N+1}} = [f_{\mathrm{Hb}}(z^*; k)]^*.
\end{aligned}
\tag{332}
$$

Part III: elliptic analytic representations

13. Basic $SU(2)$ formalism

13.1. Angular momentum operators and their polar decomposition

The unitary irreducible representations of $SU(2)$ [19–21] are labelled with the parameter j, which takes the values $j = 1/2, 1, 3/2, \ldots$. We consider the representation labelled with j and the angular momentum operators J_z, J_+, J_-:

$$
[J_z, J_+] = J_+, \qquad [J_z, J_-] = -J_-, \qquad [J_+, J_-] = 2J_z. \tag{333}
$$

The Casimir operator in this representation is

$$J^2 = J_z^2 + \tfrac{1}{2}(J_+J_- + J_-J_+) = j(j+1)\mathbf{1}. \tag{334}$$

We also consider the angular momentum states $|j; m\rangle_J$, where in agreement with our general notation we use the index 'J' to indicate that they are angular momentum states. They span a $(2j+1)$-dimensional Hilbert space $\mathcal{H}_{2j+1}$. The angular momentum operators act on the angular momentum states as follows:

$$\begin{aligned} J_+|j; m\rangle_J &= [j(j+1) - m(m+1)]^{1/2}|j; m+1\rangle_J, \\ J_-|j; m\rangle_J &= [j(j+1) - m(m-1)]^{1/2}|j; m-1\rangle_J, \\ J_z|j; m\rangle_J &= m|j; m\rangle_J, \\ J^2|j; m\rangle_J &= j(j+1)|j; m\rangle_J. \end{aligned} \tag{335}$$

We next introduce a polar decomposition of the Cartesian operators J_+ and J_- in terms of the 'radial operator' J_{r} and the '$SU(2)$ exponential of the phase operator' E [142]:

$$\begin{aligned} &J_+ = J_{\mathrm{r}}E, \qquad J_- = E^\dagger J_{\mathrm{r}}, \qquad J_{\mathrm{r}} = (J_+J_-)^{1/2}, \\ &E = \sum_m |j; m+1\rangle_J\, {}_J\langle j; m|, \\ &E^{2j+1} = \mathbf{1}, \qquad EE^\dagger = E^\dagger E = \mathbf{1}. \end{aligned} \tag{336}$$

Here $m \in \mathcal{Z}_{2j+1}$ and E is a unitary operator. We recall that in the Euclidean and hyperbolic cases the Hilbert spaces are infinite dimensional, the groups of translations and $SU(1,1)$ rotations are non-compact and the operator E_+ is isometric but non-unitary. In contrast, here the Hilbert space is finite dimensional, the $SU(2)$ group is compact and the operator E is unitary. We note that in finite-dimensional Hilbert spaces isometric operators are always unitary. We next show that

$$\begin{aligned} &J_{\mathrm{r}}|j; m\rangle_J = [j(j+1) - m(m-1)]^{1/2}|j; m\rangle_J, \\ &J_{\mathrm{r}} = \left[j(j+1)\mathbf{1} - J_z^2 + J_z\right]^{1/2} \\ &[J_{\mathrm{r}}, J_z] = 0. \end{aligned} \tag{337}$$

13.2. SU(2) transformations

We consider the unitary $SU(2)$ operators

$$\begin{aligned} &\mathcal{R}(\alpha, \beta, \gamma; j) = \exp\left[-\tfrac{1}{2}\alpha\, \mathrm{e}^{-\mathrm{i}\beta}J_+ + \tfrac{1}{2}\alpha\, \mathrm{e}^{\mathrm{i}\beta}J_-\right]\exp(\mathrm{i}\gamma J_z), \\ &0 \leqslant \alpha \leqslant \pi, \qquad 0 \leqslant \beta < 2\pi, \qquad -2\pi \leqslant \gamma < 2\pi. \end{aligned} \tag{338}$$

We note that in the case of the $SO(3)$ group which is isomorpic to $SU(2)/\mathcal{Z}_2$ the angle γ takes values $0 \leqslant \gamma < 2\pi$.

We also introduce the following notation:

$$\mathcal{R}(z; j) = \mathcal{R}(\alpha, \beta, 0; j), \qquad z = -\tan\left(\frac{\alpha}{2}\right)\mathrm{e}^{-\mathrm{i}\beta}, \tag{339}$$

where z is related to (α, β) through the stereographic projection of equation (11) and belongs in the extended complex plane C_{E}. The product of two of these operators is given by [166]

$$\mathcal{R}(z_1; j)\mathcal{R}(z_2; j) = \mathcal{R}(w; j)\exp(-\mathrm{i}\phi J_z), \tag{340}$$

where

$$w = \frac{z_1 + z_2}{1 - z_1^* z_2}, \qquad \phi = 2\arg(1 - z_1^* z_2). \tag{341}$$

This is the analogue of equation (33) in the Euclidean case and of equation (173) in the hyperbolic case.

It can also be shown that

$$\begin{aligned}
&\mathcal{R}(z;j)J_z[\mathcal{R}(z;j)]^{\dagger} = J_{\eta(z)},\\
&J_{\eta(z)} = \eta_z J_z + \frac{1}{2}(\eta_- J_+ + \eta_+ J_-),\\
&\eta_z = \frac{1-|z|^2}{1+|z|^2}, \qquad \eta_- = \eta_+^* = \frac{-2z}{1+|z|^2}.
\end{aligned} \tag{342}$$

We next prove the 'generalized resolution of the identity' property of the operators $\mathcal{R}(z;j)$. We consider an arbitrary operator $\mathcal{U}$ acting on $\mathcal{H}_{2j+1}$. We prove that

$$\int_{C_\mathrm{E}} \mathrm{d}\mu_\mathrm{S}(z)\mathcal{R}(z;j)\frac{\mathcal{U}}{\mathrm{Tr}\,\mathcal{U}}[\mathcal{R}(z;j)]^{\dagger} = \mathbf{1}. \tag{343}$$

An easy way to prove this is to use the P representation of the operator $\mathcal{U}$ which is discussed later. Acting with the operators $\mathcal{R}(z;j)$ and $[\mathcal{R}(z;j)]^{\dagger}$ on both sides equation (354), we can prove equation (343). In the special case that $\mathcal{U} = |0\rangle_{n\,n}\langle 0|$, this relation becomes the resolution of the identity for $SU(2)$ coherent states which is discussed below.

13.3. SU(2) coherent states

$SU(2)$ coherent states are defined in the coset space $SU(2)/U(1)$ which is a sphere. Through the stereographic projection of equation (11), the sphere is isomorphic to the extended complex plane C_E.

$SU(2)$ coherent states [166] are defined as

$$|z;j\rangle_\mathrm{c} = (1+|z|^2)^{-j}\sum_m d_\mathrm{S}(m;j)z^{j+m}|j;m\rangle_J, \qquad d_\mathrm{S}(m;j) = \left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2}, \tag{344}$$

where z belongs in the extended complex plane C_E. An alternative equivalent definition is

$$|z;j\rangle_\mathrm{c} = \mathcal{R}(z;j)|j;-j\rangle_J. \tag{345}$$

The equivalence between the two definitions is proved using the relation

$$\begin{aligned}
&\exp\left[-\frac{1}{2}\alpha\,\mathrm{e}^{-\mathrm{i}\beta}J_+ + \frac{1}{2}\alpha\,\mathrm{e}^{\mathrm{i}\beta}J_-\right] = \exp(zJ_+)\exp(\tau J_z)\exp(-z^*J_-),\\
&z = -\tan\left(\frac{\alpha}{2}\right)\mathrm{e}^{-\mathrm{i}\beta}, \qquad \tau = \ln(1+|z|^2).
\end{aligned} \tag{346}$$

The Baker–Campbell–Hausdorff relations for $SU(2)$ have been discussed in [140] using the general approach of [141].

Use of equation (342) leads to a third definition of $SU(2)$ coherent states as eigenstates of the Hermitian operator J_η:

$$J_{\eta(z)}|z;j\rangle_\mathrm{c} = -j|z;j\rangle_\mathrm{c}. \tag{347}$$

These states are coherent states in the sense that if we act with any of the operators $\mathcal{R}(\alpha,\beta,\gamma)$ on any of these coherent states we get another coherent state. This is because the $\mathcal{R}(\alpha,\beta,\gamma)$ operators form a group (a representation of the $SU(2)$ group).

The probability distribution

$$|{}_J\langle j;m|z;j\rangle_\mathrm{c}|^2 = \frac{(2j)!}{(j+m)!(j-m)!}\frac{|z|^{2j+2m}}{(1+|z|^2)^{2j}} \tag{348}$$

is a binomial distribution.

The overlap of two coherent states is given by

$$
{}_{\mathrm{c}}\langle z_1;\, j|z_2;\, j\rangle_{\mathrm{c}} = \left[\frac{(1+z_1^* z_2)^2}{(1+|z_1|^2)(1+|z_1|^2)}\right]^j. \tag{349}
$$

For later use, we introduce

$$
\mathcal{G}_{\mathrm{S}}(z_1, z_2;\, j) \equiv |{}_{\mathrm{c}}\langle z_1;\, j|z_2;\, j\rangle_{\mathrm{c}}|^2 = \left[\frac{|1+z_1^* z_2|^2}{(1+|z_1|^2)(1+|z_1|^2)}\right]^{2j}. \tag{350}
$$

This quantity is a function of the distance $\delta_{\mathrm{S}}(z_1, z_2)$ between the points z_1 and z_2 in spherical geometry (extended complex plane)

$$
\mathcal{G}_{\mathrm{S}}(z_1, z_2;\, j) = \left\{1 + \tan^2\left[\tfrac{1}{2}\delta_{\mathrm{S}}(z_1, z_2)\right]\right\}^{-2j}. \tag{351}
$$

The resolution of the identity in terms of $SU(2)$ coherent states is

$$
\frac{2j+1}{\pi}\int_{C_{\mathrm{E}}} |z;\, j\rangle_{\mathrm{c}\,\mathrm{c}}\langle z;\, j|\,\mathrm{d}\mu_{\mathrm{S}}(z) = \mathbf{1}. \tag{352}
$$

This shows that the set of all $SU(2)$ coherent states is at least complete. In fact, it is highly overcomplete because we will prove below that small subsets of these coherent states are also overcomplete.

13.4. P, Q and Wigner functions

The Q and P functions of an operator $\mathcal{U}$ are defined as

$$
Q_{\mathrm{S}}(z; \mathcal{U}) = {}_{\mathrm{c}}\langle z;\, j|A|z;\, j\rangle_{\mathrm{c}}, \tag{353}
$$

$$
\mathcal{U} = \frac{2j+1}{\pi}\int_{C_{\mathrm{E}}} \mathrm{d}\mu_{\mathrm{S}}(z) P_{\mathrm{S}}(z; \mathcal{U})|z;\, j\rangle_{\mathrm{c}\,\mathrm{c}}\langle z;\, j|. \tag{354}
$$

We combine these two relations and prove that

$$
Q_{\mathrm{S}}(z; \mathcal{U}) = \frac{2j+1}{\pi}\int_{C_{\mathrm{E}}} \mathrm{d}\mu_{\mathrm{S}}(z) P_{\mathrm{S}}(z; \mathcal{U})\mathcal{G}_{\mathrm{S}}(z_1, z_2;\, j). \tag{355}
$$

We also show that

$$
\mathrm{Tr}\,\mathcal{U} = \frac{2j+1}{\pi}\int_{C_{\mathrm{E}}} Q_{\mathrm{S}}(z; \mathcal{U})\,\mathrm{d}\mu_{\mathrm{S}}(z) = \frac{2j+1}{\pi}\int_{C_{\mathrm{E}}} P_{\mathrm{S}}(z; \mathcal{U})\,\mathrm{d}\mu_{\mathrm{S}}(z) \tag{356}
$$

and that

$$
\mathrm{Tr}(\mathcal{U}_1\mathcal{U}_2) = \frac{2j+1}{\pi}\int_{C_{\mathrm{E}}} Q_{\mathrm{S}}(z; \mathcal{U}_1) P_{\mathrm{S}}(z; \mathcal{U}_2)\,\mathrm{d}\mu_{\mathrm{S}}(z). \tag{357}
$$

In order to introduce Wigner functions, we first introduce the Fano [167] multipole operators

$$
T_{L,M}^{(j)} = \left(\frac{2L+1}{2j+1}\right)^{1/2} \sum_{n,m=-j}^{j} C_{j,m;L,M}^{j,n} |j;\, n\rangle_J\, {}_J\langle j;\, m|, \tag{358}
$$

where $C_{j,m;L,M}^{j,n}$ are Clebsch–Gordan coefficients

$$
C_{j,m;L,M}^{j,n} = {}_J\langle j, L;\, m, M|j;\, n\rangle_J. \tag{359}
$$

We then follow [117, 168] and define the $SU(2)$ Wigner operator

$$\mathcal{W}_{\mathrm{S}}(\alpha,\beta) = 2\left(\frac{\pi}{2j+1}\right)^{1/2}\sum_{L=0}^{2j}\sum_{M=-L}^{L} Y^*_{LM}(\alpha,\beta)T^{(j)}_{L,M}, \tag{360}$$

where $Y^*_{LM}(\alpha,\beta)$ are spherical harmonics. We can prove that

$$\mathrm{Tr}[\mathcal{W}_{\mathrm{S}}(\alpha,\beta)] = 1, \qquad \frac{2j+1}{4\pi}\int_{C_{\mathrm{E}}} \mathrm{d}\cos\alpha\,\mathrm{d}\beta\,\mathcal{W}_{\mathrm{S}}(\alpha,\beta) = \mathbf{1}. \tag{361}$$

The Wigner function of an operator $\mathcal{U}$ is defined as

$$W_{\mathrm{S}}(\alpha,\beta;\mathcal{U}) = \mathrm{Tr}[\mathcal{U}\mathcal{W}_{\mathrm{S}}(\alpha,\beta)]. \tag{362}$$

Quantum tomography methods in the context of $SU(2)$ quantum models have been discussed in [169].

14. Angle states and operators

We have considered earlier angular momentum states and operators, and $SU(2)$ coherent states. Acting on them with any unitary transformation, we can get another set of states, operators and coherent states with the same properties. Of special importance is the finite Fourier transform

$$\begin{aligned}&\mathcal{F} = (2j+1)^{-1/2}\sum_{m,n}\omega(mn)|j;m\rangle_J\,{}_J\langle j;n|, \qquad \omega(k) \equiv \exp\left(\mathrm{i}\frac{2\pi k}{2j+1}\right),\\ &\mathcal{F}^4 = \mathbf{1},\end{aligned} \tag{363}$$

where $m, n \in \mathcal{Z}_{2j+1}$.

In the harmonic oscillator formalism, the position and momentum bases (and also the position and momentum operators) are related through a Fourier transform. In analogy with this, we introduce the angle states $|j;m\rangle_\theta$ which are related to the angular momentum states $|j;m\rangle_J$ through a finite Fourier transform. We also introduce the angle operators $\theta_z, \theta_+, \theta_-$ which act on the angle states in analogous way with the angular momentum operators J_z, J_+, J_- acting on the angular momentum states.

We can then consider the θ_z–J_z angle–angular momentum phase space which is the toroidal lattice $\mathcal{Z}_{2j+1}\times\mathcal{Z}_{2j+1}$. In this phase space, we can apply the formalism of systems with finite Hilbert spaces (for a review see [69] and references therein). We can define displacement operators in $\mathcal{Z}_{2j+1}\times\mathcal{Z}_{2j+1}$, the corresponding Wigner and Weyl functions, etc. Clearly, this is very different from the displacements on a sphere in equation (339) and the Wigner functions of equation (360). In the two cases we have two different classes of transformations on different phase spaces and we associate with them different Wigner functions.

In this review, we only define the angle states and the angle operators. The phase-space formalism related to the θ_z–J_z angle–angular momentum phase space has been discussed in [69]. In the following, we distinguish the Bose sector (integer j) from the Fermi sector (half-integer j). This is because the proof of many of the formulae is based on the relation

$$\frac{1}{2j+1}\sum_n \omega[n(m-\ell)] = \delta(m,\ell), \tag{364}$$

where the summation is over all integers in $\mathcal{Z}_{2j+1}$. $\delta(m,\ell)$ is the Kronecker delta, which is equal to 1 when $m = n$ (modulo $2j+1$). This formula is valid only for integer values of m, n, ℓ. For half-integer values it is not valid, and for this reason in the Fermi sector we 'correct' by 1/2 the corresponding formulae.

14.1. Bose sector

Angle states are defined as

$$|j; m\rangle_\theta = \mathcal{F}|j; m\rangle_J = (2j+1)^{-1/2} \sum_n \omega(mn)|j; n\rangle_J \tag{365}$$

and they form an orthonormal basis in $\mathcal{H}_{2j+1}$, which is dual to the basis of angular momentum states, in the sense that the two bases are related through a Fourier transform. An arbitrary state $|f\rangle$ can be expanded in the two bases as

$$\begin{aligned} |f\rangle &= \sum_{m=-j}^{j} f_m |j; m\rangle_J = \sum_{m=-j}^{j} \tilde{f}_m |j; m\rangle_\theta, \\ f_n &= (2j+1)^{-1/2} \sum_{m=-j}^{j} \tilde{f}_m \omega(mn). \end{aligned} \tag{366}$$

f_m and $\tilde{f}_m$ are the wavefunctions of the state in the angular momentum and angle representations, respectively, and they are related through a finite Fourier transform.

Angle operators are defined as

$$\begin{aligned} &\mathcal{F}J_z\mathcal{F}^\dagger = \theta_z, \qquad \mathcal{F}J_+\mathcal{F}^\dagger = \theta_+, \qquad \mathcal{F}J_-\mathcal{F}^\dagger = \theta_-, \\ &[\theta_z, \theta_+] = \theta_+, \qquad [\theta_z, \theta_-] = -\theta_-, \qquad [\theta_+, \theta_-] = 2\theta_z, \end{aligned} \tag{367}$$

and they are generators of the $SU(2)_\theta$ group. The index 'θ' indicates the fact that from a physical point of view $SU(2)_\theta$ is different from $SU(2)$ generated by the angular momentum operators (although mathematically they are isomorphic). Both of these groups are subgroups of the $SU(2j+1)$ group of general unitary transformations in the space $\mathcal{H}_{2j+1}$. The Casimir operator is

$$\theta^2 = j(j+1)\mathbf{1} = J^2. \tag{368}$$

The angle operators act on the angle states in an analogous way to the angular momentum operators acting on the angular momentum states:

$$\begin{aligned} \theta_+|j; m\rangle_\theta &= [j(j+1) - m(m+1)]^{1/2}|j; m+1\rangle_\theta, \\ \theta_-|j; m\rangle_\theta &= [j(j+1) - m(m-1)]^{1/2}|j; m-1\rangle_\theta, \\ \theta_z|j; m\rangle_\theta &= m|j; m\rangle_\theta. \end{aligned} \tag{369}$$

General transformations in $SU(2)_\theta$ are given by

$$\mathcal{R}_\theta(\alpha, \beta, \gamma; j) = \mathcal{F}\mathcal{R}(\alpha, \beta, \gamma; j)\mathcal{F}^\dagger = \exp\left[-\tfrac{1}{2}\alpha\, e^{-i\beta}\theta_+ + \tfrac{1}{2}\alpha\, e^{i\beta}\theta_-\right]\exp(i\gamma\theta_z). \tag{370}$$

θ-coherent states are defined as

$$|z; j\rangle_{c\theta} = \mathcal{F}|z; j\rangle_c = (1+|z|^2)^{-j} \sum_n d_S(m, j) z^{j+n} |j; m\rangle_\theta. \tag{371}$$

The index '$c\theta$' is used in the notation of these states.

We next consider a polar decomposition of θ_+ and θ_- similar to that given in equation (336) for the angular momentum operators:

$$\begin{aligned} &\theta_+ = \theta_r G, \qquad \theta_- = G^\dagger\theta_r, \\ &\theta_r = (\theta_+\theta_-)^{1/2} = \mathcal{F}J_r\mathcal{F}^\dagger, \\ &G = \mathcal{F}E\mathcal{F}^\dagger = \sum_m |j; m+1\rangle_{\theta\,\theta}\langle j; m|. \end{aligned} \tag{372}$$

Here the 'radial operator' is θ_r and the 'exponential of the phase operator' is G. We can show that operator G is the exponential of J_z, and similarly that the operator E is the exponential of θ_z:

$$E = \exp\left(-\mathrm{i}\frac{2\pi}{2j+1}\theta_z\right), \qquad G = \exp\left(\mathrm{i}\frac{2\pi}{2j+1}J_z\right). \tag{373}$$

The operators E and G displace the states $|j;m\rangle_J$ and $|j;m\rangle_\theta$ in the θ_z–J_z angle–angular momentum phase space which is the toroidal lattice $\mathcal{Z}_{2j+1} \times \mathcal{Z}_{2j+1}$:

$$\begin{aligned} &G^n|j;m\rangle_\theta = |j;m+n\rangle_\theta, && G^n|j;m\rangle_J = \omega(nm)|j;m\rangle_J,\\ &E^\ell|j;m\rangle_\theta = \omega(-m\ell)|j;m\rangle_\theta, && E^\ell|j;m\rangle_J = |j;m+\ell\rangle_J, \end{aligned} \tag{374}$$

where $n, \ell \in \mathcal{Z}_{2j+1}$. The operators E and G form a Heisenberg–Weyl group:

$$E^\ell G^n = G^n E^\ell \omega(-n\ell), \qquad E^{2j+1} = G^{2j+1} = \mathbf{1}. \tag{375}$$

This is the starting point of a phase-space formalism in the θ_z–J_z angle–angular momentum phase space [69].

14.2. Fermi sector

The Fourier operator in the Fermi sector is given by

$$\mathcal{F} = (2j+1)^{-1/2}\sum_{m,n}\omega\left[\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\right]|j;m\rangle_{J\,J}\langle j;n|. \tag{376}$$

m, n take half-integer values and therefore $m+1/2$, $n+1/2$ take integer values, so that we can use equation (364). The angle states are defined as

$$|j;m\rangle_\theta = \mathcal{F}|j;m\rangle_J = (2j+1)^{-1/2}\sum_n \omega\left[\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\right]|j;n\rangle_J. \tag{377}$$

Equations (367)–(372) and also equation (375) are the same in both Bose and Fermi sectors. Equations (373) and (374) in the Fermi sector are given by

$$\begin{aligned} &E = \exp\left[-\mathrm{i}\frac{2\pi}{2j+1}\left(\theta_z+\frac{1}{2}\right)\right], && G = \exp\left[\mathrm{i}\frac{2\pi}{2j+1}\left(J_z+\frac{1}{2}\right)\right],\\ &G^n|j;m\rangle_\theta = |j;m+n\rangle_\theta, && G^n|jm\rangle_J = \omega\left[n\left(m+\frac{1}{2}\right)\right]|j;m\rangle_J,\\ &E^\ell|j;m\rangle_\theta = \omega\left[-\ell\left(m+\frac{1}{2}\right)\right]|j;m\rangle_\theta, && E^\ell|j;m\rangle_J = |j;m+\ell\rangle_J, \end{aligned} \tag{378}$$

where n, ℓ are integers in $\mathcal{Z}_{2j+1}$.

15. The $SU(2)$ formalism in the harmonic oscillator context

15.1. The Holstein–Primakoff $SU(2)$ formalism

The Holstein–Primakoff $SU(2)$ formalism [170] considers a $(2j+1)$-dimensional subspace $\mathcal{H}_{\mathrm{tr}}$ of the harmonic oscillator Hilbert space $\mathcal{H}$. The space $\mathcal{H}_{\mathrm{tr}}$ (where the index 'tr' stands for truncated) is spanned by the number states $|N\rangle_n$ with $0 \leqslant N \leqslant 2j$. In $\mathcal{H}_{\mathrm{tr}}$ we consider the operators

$$\begin{aligned} &J_+ = [(2j+1) - a^\dagger a]^{1/2}a^\dagger, && J_- = a[(2j+1) - a^\dagger a]^{1/2},\\ &J_z = a^\dagger a - j && J^2 = j(j+1)\mathbf{1}. \end{aligned} \tag{379}$$

Here **1** is the unit operator in $\mathcal{H}_{\mathrm{tr}}$. These operators obey the usual angular momentum commutation relations. The corresponding angular momentum states are the number eigenstates:

$$|j; m\rangle_J = |N\rangle_n, \qquad N = j + m. \tag{380}$$

The Holstein–Primakoff $SU(2)$ formalism expresses the angular momentum operators in terms of harmonic oscillator creation and annihilation operators. It uses a single harmonic oscillator in contrast to the Schwinger formalism which uses two harmonic oscillators, and which is discussed below.

15.2. The Schwinger SU(2) formalism

The Schwinger representation of $SU(2)$ involves two harmonic oscillators and is given by

$$J_+ = a_1^\dagger a_2, \qquad J_- = a_1 a_2^\dagger, \qquad J_z = \tfrac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right). \tag{381}$$

We can easily check that they obey the $SU(2)$ commutation relations. If n_s is the sum of the number operators of the two oscillators, then the Casimir operator is given by

$$\begin{aligned} &J^2 = \frac{n_s}{2}\left(\frac{n_s}{2}+1\right), \qquad n_s = a_1^\dagger a_1 + a_2^\dagger a_2, \\ &[n_s, J_+] = [n_s, J_-] = [n_s, J_z] = 0. \end{aligned} \tag{382}$$

Let $\mathcal{H}_1 \times \mathcal{H}_2$ be the Hilbert space describing the two oscillators. We consider its subspaces

$$\mathcal{H}_{2j+1} = \mathrm{span}\{|2j - N_2, N_2\rangle_n; 0 \leqslant N_2 \leqslant 2j\}. \tag{383}$$

It is clear that the sum of the average number of photons in the two oscillators, for all states in $\mathcal{H}_{2j+1}$, is equal to $2j + 1$. We act on the states of $\mathcal{H}_{2j+1}$ with the operators of equation (381) and (382) and we get

$$\begin{aligned} &J_z|2j - N_2, N_2\rangle_n = (j - N_2)|2j - N_2, N_2\rangle_n, \\ &J^2|2j - N_2, N_2\rangle_n = j(j+1)|2j - N_2, N_2\rangle_n. \end{aligned} \tag{384}$$

Comparison with the general relations for the $SU(2)$ representations given in equation (335) shows that the operators of equation (381) and (382) acting on $\mathcal{H}_{2j+1}$ form the j representation. It also shows that the harmonic oscillator number states $|N_1, N_2\rangle_n$ are also the angular momentum states:

$$|N_1, N_2\rangle_n = |j; m\rangle_J, \qquad j = \tfrac{1}{2}(N_1 + N_2), \qquad m = \tfrac{1}{2}(N_1 - N_2). \tag{385}$$

The full Hilbert space $\mathcal{H}_1 \times \mathcal{H}_2$ is the direct sum of all the spaces $\mathcal{H}_{2j+1}$ (for all integer and half-integer j). We call Π_{2j+1} the projection operators to the spaces $\mathcal{H}_{2j+1}$. It is clear that

$$[\Pi_{2j+1}, J_z] = [\Pi_{2j+1}, J_+] = [\Pi_{2j+1}, J_-] = [\Pi_{2j+1}, n_s] = 0. \tag{386}$$

Therefore, any operator which is a function of the angular momentum operators leaves the Hilbert spaces $\mathcal{H}_{2j+1}$ invariant (i.e., acting on a state in $\mathcal{H}(2j+1)$ produces another state which belongs in the same Hilbert space).

As an application, we consider the following Hamiltonian which is used for the description of frequency converters in quantum optics [171]:

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \lambda a_1^\dagger a_2 + \lambda^* a_1 a_2^\dagger. \tag{387}$$

It can be written in terms of the $SU(2)$ generators of equations (381) and (382) as

$$H = \tfrac{1}{2}(\omega_1 + \omega_2)n_s + (\omega_1 - \omega_2)J_z + \lambda J_+ + \lambda^* J_-. \tag{388}$$

The time evolution of these systems can be studied using the analytic methods discussed below.

16. Analytic representations in the extended complex plane based on $SU(2)$ coherent states

16.1. States

We consider a state $|f\rangle$ in the $(2j+1)$-dimensional space $\mathcal{H}_{2j+1}$

$$|f\rangle = \sum_{m=-j}^{j} f_m |j;m\rangle_J, \qquad \sum_{m=-j}^{j} |f_m|^2 = 1. \tag{389}$$

This state is represented by the polynomial

$$f_S(z;j) = \sum_{m=-j}^{j} d_S(m;j) f_m z^{j+m} = (1+|z|^2)^j \ {}_c\langle z^*;j|f\rangle = (1+|z|^2)^j \langle f^*|z;j\rangle_c, \tag{390}$$

where the coefficients $d_S(m;j)$ have been defined in equation (344). The order of this polynomial is less or equal to $2j$. $f_S(z;j)$ is analytic in the extended complex plane.

Let ℓ_0 be an anticlockwise contour around the origin. Then

$$\oint_{\ell_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{f_S(z;j)}{d_S(m;j) z^{j+m+1}} = f_m. \tag{391}$$

This integral gives the coefficients f_m. Relations analogous to equations (92) and (93) also hold here.

The scalar product of two states $|f\rangle$ and $|g\rangle$ represented by the functions $f_S(z;j)$ and $g_S(z;j)$, respectively, is given by

$$\langle g|f\rangle = \frac{2j+1}{\pi} \int_{C_E} [g_S(z;j)]^* f_S(z;j)(1+|z|^2)^{-2j}\, \mathrm{d}\mu_S(z). \tag{392}$$

As examples, we consider the angular momentum states and the $SU(2)$ coherent states

$$|j;m\rangle_J \to d_S(m;j) z^{j+m}, \qquad |\zeta;j\rangle_c \to \frac{(1+z\zeta)^{2j}}{(1+|\zeta|^2)^j}. \tag{393}$$

For the θ-coherent states of equation (371), we get

$$|\zeta;j\rangle_{c\theta} \to (2j+1)^{-1/2}(1+|\zeta|^2)^{-j} \sum_{n,m} \zeta^{j+m} z^{j+n} \omega(nm). \tag{394}$$

The angular momentum operators are represented as

$$J_+ = -z^2\partial_z + 2jz, \qquad J_- = \partial_z, \qquad J_z = z\partial_z - j. \tag{395}$$

We can easily check that these operators acting on $d_S(m;j)z^{j+m}$ give the expected results of equations (335).

$SU(2)$ transformations in the extended complex plane are implemented through the Möbius conformal mappings

$$w(z) = \frac{\kappa z - \lambda^*}{\lambda z + \kappa^*}, \qquad |\kappa|^2 + |\lambda|^2 = 1. \tag{396}$$

Then the transformations

$$|f\rangle \to \mathcal{R}(\alpha,\beta,\gamma;j)|f\rangle \tag{397}$$

are implemented as

$$f_S(z;j) \to f_S\left(\frac{\kappa z - \lambda^*}{\lambda z + \kappa^*};j\right)(\lambda z + \kappa^*)^{2j} = \sum_n d_S(n;j) f_n [\kappa z - \lambda^*]^{j+n} [\lambda z + \kappa^*]^{j-n}. \tag{398}$$

The relation between the complex coefficients κ and λ and the variables α, β, γ is

$$\kappa = \cos\left(\frac{\alpha}{2}\right)\mathrm{e}^{\mathrm{i}\gamma/2}, \qquad \lambda = -\sin\left(\frac{\alpha}{2}\right)\mathrm{e}^{-\mathrm{i}\beta}. \tag{399}$$

The term $(\lambda z + \kappa^*)^{2j}$ is the 'multiplier'. Similar term appears in the hyperbolic case of equation (249).

The infinitesimal version of the mapping of equation (396) is

$$w = z + \epsilon_- + \epsilon_0 z - \epsilon_+ z^2, \tag{400}$$

where $\epsilon_-, \epsilon_0, \epsilon_+$ are infinitesimals. In this case

$$f_{\mathrm{S}}(z; j) \rightarrow [1 + \epsilon_- J_- + \epsilon_0 J_z + \epsilon_+ J_+] f_{\mathrm{S}}(z; j), \tag{401}$$

where J_0, J_-, J_+ are the differential operators of equation (395). The term $2jz$ in the operator J_+ is an infinitesimal multiplier. The same is true for the term $-j$ in the operator J_z.

Let r be the order of the polynomial $f_{\mathrm{S}}(z; j)$. The number of zeros of this polynomial is also r. Clearly, $r \leqslant 2j$. We note that here the zeros define uniquely the state. In contrast, in the Euclidean and hyperbolic cases where the Hilbert spaces are infinite dimensional, the zeros do not define uniquely the state.

If w is a zero of the function $f_{\mathrm{S}}(z; j)$, then the state $|f^*\rangle$ is orthogonal to the coherent state $|w; j\rangle_{\mathrm{c}}$. A more general result is that if w is a zero of multiplicity M then the state $|f^*\rangle$ is orthogonal to all states $(J_+)^N |w; j\rangle_{\mathrm{c}}$ where $N = 0, \ldots, M-1$. The proof is analogous to equation (125) for the Euclidean case.

Any set of more than $2j SU(2)$ coherent states is at least complete. Indeed, if this is not a complete set then there exist a state $|f^*\rangle$ which is orthogonal to all of them. Then the corresponding polynomial $f_{\mathrm{S}}(z; j)$ is of order $2j$ and has more than $2j$ zeros, which is not possible.

16.2. Operators

An arbitrary operator $\mathcal{U}$ with matrix elements $\mathcal{U}_{nm} = {}_J\langle n; j|\mathcal{U}|m; j\rangle_J$ is represented by the following kernel:

$$\begin{aligned} \mathcal{K}_{\mathrm{S}}(z, \zeta^*; \mathcal{U}) &\equiv (1 + |z|^2)^j (1 + |\zeta|^2)^j \, {}_{\mathrm{c}}\langle z^*|\mathcal{U}|\zeta^*\rangle_{\mathrm{c}} \\ &= \sum_{n,m} [d_{\mathrm{S}}(n; j) z^{j+n}] \mathcal{U}_{nm} [d_{\mathrm{S}}(m; j)(\zeta^*)^{j+m}] \end{aligned} \tag{402}$$

and the state $|g\rangle = \mathcal{U}|f\rangle$ is represented by the function

$$g_{\mathrm{S}}(z; j) = \frac{2j+1}{\pi} \int_{C_{\mathrm{E}}} \mathcal{K}_{\mathrm{S}}(z, \zeta^*; \mathcal{U}) f_{\mathrm{S}}(\zeta; j)(1 + |\zeta|^2)^{-2j} \, \mathrm{d}\mu_{\mathrm{S}}(\zeta). \tag{403}$$

The kernel $\mathcal{K}_{\mathrm{S}}(z_1, z_2^*; \mathcal{U})$ is an analytic function of z and ζ^* in the extended complex plane. Consequently, its diagonal component $\mathcal{K}_{\mathrm{S}}(z, z^*; \mathcal{U})$ determines uniquely through analytic continuation $\mathcal{K}_{\mathrm{S}}(z_1, z_2^*; \mathcal{U})$.

As an example, we consider the unit operator for which

$$\mathcal{K}_{\mathrm{S}}(z, \zeta^*; \mathbf{1}) = (1 + z\zeta^*)^{2j}. \tag{404}$$

This is the 'reproducing kernel'. For any state $|f\rangle$, the $f_{\mathrm{S}}(z; j)$ is related to $f_{\mathrm{S}}(\zeta; j)$ through the relation

$$f_{\mathrm{S}}(z; j) = \frac{2j+1}{\pi} \int_{C_{\mathrm{E}}} \frac{(1 + z\zeta^*)^{2j}}{(1 + |\zeta|^2)^{2j}} f_{\mathrm{S}}(\zeta; j) \, \mathrm{d}\mu_{\mathrm{S}}(\zeta). \tag{405}$$

As a second example, we consider the angular momentum operator J_z for which

$$\mathcal{K}_{\mathrm{S}}(z, \zeta^*; J_z) = -j(1 + z\zeta^*)^{2j-1}(1 - z\zeta^*). \tag{406}$$

It can be shown that this representation of J_z with a kernel is consistent with the differential form of J_z given in equation (395).

The trace of an operator $\mathcal{U}$ is given by

$$\mathrm{Tr}\,\mathcal{U} = \frac{2j+1}{\pi}\int_{C_{\mathrm{E}}} \mathcal{K}_{\mathrm{S}}(z,z^*;\mathcal{U})(1+|z|^2)^{-2j}\,\mathrm{d}\mu_{\mathrm{S}}(z). \tag{407}$$

The product of two operators $\mathcal{U}_1\mathcal{U}_2$ is represented with the kernel

$$\mathcal{K}_{\mathrm{S}}(z,\zeta^*;\mathcal{U}_1\mathcal{U}_2) = \frac{2j+1}{\pi}\int_{C_{\mathrm{E}}} \mathcal{K}_{\mathrm{S}}(z,w^*;\mathcal{U}_1)\mathcal{K}_{\mathrm{S}}(w,\zeta^*;\mathcal{U}_2)(1+|w|^2)^{-2j}\,\mathrm{d}\mu_{\mathrm{S}}(w). \tag{408}$$

We note that we have some freedom to define slightly different kernels for the representation of the operators, provided that we modify accordingly the above relations. Below we will use the following kernel

$$\mathcal{L}_{\mathrm{S}}(z,\zeta^*;\mathcal{U}) = \mathcal{K}_{\mathrm{S}}(z,\zeta^*;\mathcal{U})(1+z\zeta^*)^{-2j}. \tag{409}$$

16.3. Elliptic Berezin formalism

We represent the various operators with the $\mathcal{L}$-kernels of equation (409). Using equation (408) we show that the diagonal part of the kernel representing the product $\mathcal{U}_1\mathcal{U}_2$ of two operators is given by

$$\mathcal{L}_{\mathrm{S}}(z,z^*;\mathcal{U}_1\mathcal{U}_2) = \frac{2j+1}{\pi}\int_{C_{\mathrm{E}}} \mathcal{G}_{\mathrm{S}}(z_1,z_2;j)\mathcal{L}_{\mathrm{S}}(z,w^*;\mathcal{U}_1)\mathcal{L}_{\mathrm{S}}(w,z^*;\mathcal{U}_2)\,\mathrm{d}\mu_{\mathrm{S}}(w), \tag{410}$$

where $\mathcal{G}_{\mathrm{S}}(z_1,z_2;j)$ has been given in equation (350). Berezin [30] has proved that

$$\frac{2j+1}{\pi}\int_{C_{\mathrm{E}}} \mathcal{G}_{\mathrm{S}}(z_1,z_2;j)f(w,w^*)\,\mathrm{d}\mu_{\mathrm{S}}(w) = \chi\left(\Delta_z^{(\mathrm{S})}\right)f(z,z^*), \tag{411}$$

where the elliptic Laplace–Beltrami operator has been given in equation (14) and

$$\chi\left(\Delta_z^{(\mathrm{S})}\right) = \prod_{N=1}^{\infty}\left(1+\lambda_N\Delta_z^{(\mathrm{S})}\right), \qquad \lambda_N = \frac{1}{2j+N} - \frac{1}{2j+N+1}. \tag{412}$$

This is the elliptic analogue of equation (85) in the Euclidean case and equation (262) in the hyperbolic case. Therefore, equation (410) can be written as

$$\mathcal{L}_{\mathrm{S}}(z,z^*;\mathcal{U}_1\mathcal{U}_2) = \left[\chi\left(\Delta_\zeta^{(\mathrm{S})}\right)\mathcal{L}_{\mathrm{S}}(z,\zeta^*;\mathcal{U}_1)\mathcal{L}_{\mathrm{S}}(\zeta,z^*;\mathcal{U}_2)\right]_{\zeta=z}. \tag{413}$$

In the limit $j\to\infty$, we get

$$\chi\left(\Delta_z^{(\mathrm{S})}\right) = 1 + \frac{1}{2j+1}\Delta_z^{(\mathrm{S})} + \cdots. \tag{414}$$

Therefore,

$$\mathcal{L}_{\mathrm{S}}(z,z^*;\mathcal{U}_1\mathcal{U}_2) = \mathcal{L}_{\mathrm{S}}(z,z^*;\mathcal{U}_1)\mathcal{L}_{\mathrm{S}}(z,z^*;\mathcal{U}_2) + \frac{(1+|z|^2)}{2j+1}\frac{\partial\mathcal{L}_{\mathrm{S}}(z,z^*;\mathcal{U}_1)}{\partial z^*}\frac{\partial\mathcal{L}_{\mathrm{S}}(z,z^*;\mathcal{U}_2)}{\partial z} + \cdots. \tag{415}$$

The parameter $1/j$ plays similar role to the Planck constant. In the semiclassical limit $j\to\infty$, only the first term survives and we get the classical result that the operators commute:

$$\mathcal{L}_{\mathrm{S}}(z,z^*;\mathcal{U}_1\mathcal{U}_2) = \mathcal{L}_{\mathrm{S}}(z,z^*;\mathcal{U}_1)\mathcal{L}_{\mathrm{S}}(z,z^*;\mathcal{U}_2). \tag{416}$$

We next keep the first two terms in the expansion of equation (415) and show that that the commutator $[\mathcal{U}_1,\mathcal{U}_2]$ is represented by the function

$$\mathcal{L}_{\mathrm{S}}(z,z^*;[\mathcal{U}_1,\mathcal{U}_2]) = -\frac{\mathrm{i}}{2j+1}\{\mathcal{L}_{\mathrm{S}}(z,z^*;\mathcal{U}_1),\mathcal{L}_{\mathrm{S}}(z,z^*;\mathcal{U}_2)\}_{\mathrm{S}}, \tag{417}$$

where the elliptic Poisson bracket has been given in equation (13). It is seen that when the higher order terms in the above $1/j$ expansion are turned off, the quantum mechanical commutator reduces to the elliptic Poisson bracket.

As we have already mentioned in the Euclidean and hyperbolic cases, these semiclassical expressions assume that the relevant functions are smooth functions of $1/j$ so that the expansion of equation (415) is valid.

17. Elliptic contour representation in the extended complex plane

17.1. States

In the elliptic contour formalism [51], the arbitrary state $|f\rangle$ of equation (389) and the corresponding bra state $\langle f|$ are represented as follows:

$$\begin{aligned} |f\rangle &\rightarrow f_{\mathrm{Sk}}(z;j) = \sum_n d_{\mathrm{S}}(n;j) f_n z^{j+n}, \\ \langle f| &\rightarrow f_{\mathrm{Sb}}(z;j) = \sum_n \frac{f_n^*}{d_{\mathrm{S}}(n;j) z^{j+n+1}}. \end{aligned} \tag{418}$$

We use the indices 'Sk' and 'Sb' to indicate ket and bra states for the elliptic case, respectively. The ket function $f_{\mathrm{Sk}}(z;j)$ is the same with the function $f_{\mathrm{S}}(z;j)$ of equation (390). It is a polynomial of order less or equal to $2j$ and has singularity at ∞ (which is the north pole). The bra function $f_{\mathrm{Sb}}(z;j)$ is a polynomial of z^{-1} of order $2j+1$ and has singularity at 0 (which is the south pole). Clearly, the convergence difficulties which we had in the Euclidean and hyperbolic cases do not appear here.

As examples, we consider the angular momentum states $|j;m\rangle_J$ for which

$$f_{\mathrm{Sk}}(z;j) = d_{\mathrm{S}}(m;j) z^{j+m}, \qquad f_{\mathrm{Sb}}(z) = \frac{1}{d_{\mathrm{S}}(m;j) z^{j+m+1}} \tag{419}$$

and the coherent states $|\zeta;j\rangle_{\mathrm{c}}$ for which

$$f_{\mathrm{Sk}}(z;j) = \frac{(1+z\zeta)^{2j}}{(1+|\zeta|^2)^j}, \qquad f_{\mathrm{Sb}}(z;j) = \frac{1}{(1+|\zeta|^2)^j} \frac{z^{2j+1} - (\zeta^*)^{2j+1}}{(z-\zeta^*) z^{2j+1}}. \tag{420}$$

We note that $f_{\mathrm{Sb}}(z;j)$ corresponding to the coherent state has a singularity at $z=0$ but has no singularity at $z=\zeta^*$.

The scalar product is given by

$$\langle f|g\rangle = \oint_\ell \frac{\mathrm{d}z}{2\pi\mathrm{i}} f_{\mathrm{Sb}}(z;j) g_{\mathrm{Sk}}(z;j), \tag{421}$$

where ℓ is an anticlockwise contour around the origin.

The following transformations connect the bra with the ket function:

$$\oint_\ell \frac{\mathrm{d}w}{2\pi\mathrm{i}} f_{\mathrm{Sb}}(w;j)(1+z^*w)^{2j} = [f_{\mathrm{Sk}}(z;j)]^*, \tag{422}$$

$$\frac{2j+1}{z} \int_0^\infty \frac{\mathrm{d}t}{(1+t)^{2j+2}} \left[f_{\mathrm{Sk}}\left(\frac{t}{z^*}\right)\right]^* = f_{\mathrm{Sb}}(z;j). \tag{423}$$

17.2. Operators

An arbitrary operator $\mathcal{U}$ with matrix elements $\mathcal{U}_{MN} = {}_J\langle j; m|\mathcal{U}|j; n\rangle_J$ is represented by the kernel

$$\mathcal{C}_{\mathrm{S}}(z_1, z_2; \mathcal{U}) = \sum_{n,m} \mathcal{U}_{nm} \frac{d_{\mathrm{S}}(n; j) z_1^{j+n}}{d_{\mathrm{S}}(m; j) z_2^{j+m+1}}. \tag{424}$$

As examples, we give the kernels for the operators $\mathbf{1}$, J_z:

$$\begin{aligned} \mathcal{C}_{\mathrm{S}}(z_1, z_2; \mathbf{1}) &= \frac{z_1^{2j+1} - z_2^{2j+1}}{(z_1 - z_2) z_2^{2j+1}}, \\ \mathcal{C}_{\mathrm{S}}(z_1, z_2; J_z) &= \frac{j\big(z_1^{2j+2} - z_2^{2j+2}\big) - (j+1) z_1 z_2 \big(z_1^{2j} - z_2^{2j}\big)}{\big(z_1^2 - z_2^2\big) z_2^{2j+1}}. \end{aligned} \tag{425}$$

The ket state $|g\rangle = \mathcal{U}|f\rangle$ and the bra state $\langle g| = \langle f|\mathcal{U}^\dagger$ are represented as follows:

$$\begin{aligned} g_{\mathrm{Sk}}(z_1) &= \oint_\ell \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \mathcal{C}_{\mathrm{S}}(z_1, z_2; \mathcal{U}) f_{\mathrm{Sk}}(z_2), \\ g_{\mathrm{Sb}}(z_2) &= \oint_\ell \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} f_{\mathrm{Sb}}(z_1) \mathcal{C}_{\mathrm{S}}(z_1, z_2; \mathcal{U}^\dagger). \end{aligned} \tag{426}$$

We have explained earlier that any set of more than $2j$ $SU(2)$ coherent states is at least complete. Therefore, the coherent states on a contour form a highly overcomplete set of states. The elliptic contour formalism can be used to expand an arbitrary state $|f\rangle$ in terms of $SU(2)$ coherent states on a contour ℓ around the origin [51]:

$$|f\rangle = \oint_\ell \frac{\mathrm{d}z}{2\pi\mathrm{i}} (1 + |z|^2)^j a(z) |z; j\rangle_{\mathrm{c}}, \qquad a(z) = \sum_n \frac{f_n}{d_{\mathrm{S}}(n; j) z^{j+n+1}} = [f_{\mathrm{Sb}}(z^*; j)]^*. \tag{427}$$

Part IV: analytic representations on a torus

18. Analytic representation of finite quantum systems in terms of theta functions on a torus

In the harmonic oscillator both position and momentum take values in R and the phase space is the plane $R \times R$.

Quantum mechanics on a circle T has also been studied in the literature [172, 173]. For quasi-periodic boundary conditions ($f(x + 2\pi) = f(x)\,\mathrm{e}^{\mathrm{i}\theta}$), we get Aharonov–Bohm phenomena [174]. In this case, the momentum takes values in $\mathcal{Z}$ (integer plus $\theta/2\pi$) and the phase space is $T \times \mathcal{Z}$. Coherent states on a circle have been studied in [175], and Wigner functions in [176].

Quantum systems with finite-dimensional Hilbert space have been studied extensively in the literature (for a review see [69] and references therein). A system with angular momentum j is an example of such system with dimension of the Hilbert space $2j+1$. We have explained earlier that in this case both the angular position θ_z and the angular momentum J_z take values in $\mathcal{Z}_{2j+1}$ and the phase space is the toroidal lattice $\mathcal{Z}_{2j+1} \times \mathcal{Z}_{2j+1}$. We have defined the elliptic analytic representation of equation (390) which is particularly suitable for the study of $SU(2)$ rotations in these systems.

Here we define a different analytic representation for these systems. We first use a special case of the Zak transform [177] to introduce a map between states in the infinite-dimensional harmonic oscillator Hilbert space $\mathcal{H}$ and states in the $(2j+1)$-dimensional Hilbert space

$\mathcal{H}_{2j+1}$. Using this map, we apply ideas from the harmonic oscillator formalism to systems with finite Hilbert space $\mathcal{H}_{2j+1}$. For example, the coherent (Gaussian) states of the harmonic oscillator will be mapped to theta functions [103, 178] in $\mathcal{H}_{2j+1}$. Consequently, the Bargmann formalism for the harmonic oscillator which is based on coherent states will become an analytic formalism based on theta functions. This formalism uses entire functions which obey quasi-periodic boundary conditions and therefore it is sufficient to define them on a square cell S, i.e. on a torus. All these functions have exactly $2j+1$ zeros in each cell S, and if the zeros are given we can construct the analytic representation of the state [179].

We note that we can introduce directly the analytic representation of equation (440) below, without any reference to the Zak transform. We believe, however, that it is constructive to understand the relation between the harmonic oscillator formalism and the finite quantum systems formalism through the Zak transform given below in equation (428), and to see how the Gaussian functions become theta functions.

We have mentioned earlier in connection with equation (148) that theta functions have been introduced in [1, 135] through the study of the eigenvectors of the displacement operators $D(z)$ in a von Neumann lattice with cell area π (which belong in an extended space). Here we introduce theta functions in finite systems which are however related to the harmonic oscillator through the Zak transform of equation (428). The relation between the two approaches requires further study. Other work on phase-space methods with theta functions has been reported in [180].

We have explained earlier that some of the formulae are different in the Bose (integer j) and Fermi (half-integer j) cases, and below we consider the Bose case.

18.1. Zak transform

We consider a state $|g\rangle$ in the harmonic oscillator Hilbert space $\mathcal{H}$ with wavefunctions $g_x(x)$ and $g_p(p)$ in the x and p representations, respectively. To this state we map another state $|f\rangle$ in $\mathcal{H}_{2j+1}$ which is defined as

$$\begin{aligned} |f\rangle &= \sum_m f_m |j; m\rangle_J, \\ f_m &= \mathcal{N}^{-1/2} \sum_{w=-\infty}^{\infty} g_x[2^{1/2}\pi m A^{-1/2} + 2^{1/2} w A^{1/2}] \\ &= \mathcal{N}^{-1/2}(2j+1)^{-1/2} \sum_{w=-\infty}^{\infty} g_p(2^{1/2}\pi w A^{-1/2})\omega(mw), \\ A &= \pi(2j+1), \end{aligned} \tag{428}$$

where $m \in \mathcal{Z}_{2j+1}$. This map is a special case of the Zak transform. The normalization factor $\mathcal{N}$ is such that $\sum |f_m|^2 = 1$. In order to prove equation (428), we use the Poisson formula

$$\sum_{w=-\infty}^{\infty} \exp(\mathrm{i}2\pi wx) = \sum_{k=-\infty}^{\infty} \delta(x-k). \tag{429}$$

The right-hand side is the 'comb delta function'. The state of equation (428) can also be expanded in the basis of angle states as

$$|f\rangle = \sum_m \tilde{f}_m |j; m\rangle_\theta, \qquad \tilde{f}_m = (2j+1)^{-1/2} \sum_n f_n \omega(-mn). \tag{430}$$

We can show that $\tilde{f}_m$ is given in terms of the wavefunctions $g_x(x)$, $g_p(p)$ as

$$\begin{aligned}\tilde{f}_m &= \mathcal{N}^{-1/2}\sum_{w=-\infty}^{\infty} g_p[2^{1/2}\pi m A^{-1/2} + 2^{1/2} w A^{1/2}]\\ &= \mathcal{N}^{-1/2}(2j+1)^{-1/2}\sum_{w=-\infty}^{\infty} g_x(2^{1/2}\pi w A^{-1/2})\omega(-mw).\end{aligned} \tag{431}$$

It is easily seen that

$$f_{m+2j+1} = f_m, \qquad \tilde{f}_{m+2j+1} = \tilde{f}_m. \tag{432}$$

The coefficients f_m and $\tilde{f}_m$ are related through the finite Fourier transform of equation (366). At the same time, equations (428) and (431) show that they are the Zak transforms of g_x and g_p which are related through the Fourier transform of equation (17), in $\mathcal{H}$. Therefore, the Fourier transform in $\mathcal{H}$ becomes through the Zak transform the finite Fourier transform in $\mathcal{H}_{2j+1}$.

The above map is not one-to-one and equation (428) cannot be inverted. We note that we can use the full Zak transform and introduce a family of d-dimensional Hilbert spaces $\mathcal{H}_{2j+1}(\sigma_1, \sigma_2)$ where $0 \leqslant \sigma_1 < 1$ and $0 \leqslant \sigma_2 < 1$. These Hilbert spaces have twisted boundary conditions [44] and they are spanned by the states corresponding to $f_m(\sigma_1, \sigma_2)$ where

$$f_m(\sigma_1, \sigma_2) = [\mathcal{N}(\sigma_1, \sigma_2)]^{-1/2}\sum_{w=-\infty}^{\infty}\exp(-\mathrm{i}2\pi\sigma_1 w) g_x\{[m + (2j+1)w + \sigma_2]2^{1/2}\pi A^{-1/2}\}. \tag{433}$$

The harmonic oscillator Hilbert space $\mathcal{H}$ is isomorphic to the direct integral of all the spaces $\mathcal{H}_{2j+1}(\sigma_1, \sigma_2)$. In this case, an inverse to the relation (433) can be found. Here we limit ourselves to the map of equation (428), which is a special case of the map (433) with $\sigma_1 = \sigma_2 = 0$.

As an example, we consider the Glauber coherent states $|z\rangle_{\mathrm{c}}$ of equation (64) in $\mathcal{H}$ and using equation (428) we get the corresponding 'coherent states' $|f\rangle_{\mathrm{cf}}$ in $\mathcal{H}_{2j+1}$ [42, 179, 181]. The index 'cf' stands for 'coherent and finite' and is used in order to distinguish these states from the harmonic oscillator coherent states. They are given by

$$\begin{aligned}|z\rangle_{\mathrm{cf}} &= \sum_m s_m(z)|j;m\rangle_J,\\ s_m(z) &= [\mathcal{N}(z)]^{-1/2}\pi^{-1/4}\exp[-\pi^2 m^2 A^{-1} + 2\pi z m A^{-1/2} - z_{\mathrm{R}} z]\\ &\quad\times\Theta_3[-\mathrm{i}\pi m + \mathrm{i} z A^{1/2}; \mathrm{i}(2j+1)]\\ &= [\mathcal{N}(z)]^{-1/2}\pi^{1/4}A^{-1/2}\,\mathrm{e}^{\mathrm{i}z_{\mathrm{I}}z}\Theta_3\left[-\pi^2 m A^{-1} + \pi z A^{-1/2}; \frac{\mathrm{i}}{2j+1}\right],\end{aligned} \tag{434}$$

where Θ_3 are theta functions defined as [103]

$$\Theta_3(u;\tau) = \sum_{n=-\infty}^{\infty}\exp(\mathrm{i}\pi\tau n^2 + \mathrm{i}2nu). \tag{435}$$

The normalization factors $\mathcal{N}(z)$ have been given in [179], for the bosonic (integer j) and fermionic (half-integer j) cases. The coefficients $s_m(z)$ obey the quasi-periodicity relations

$$\begin{aligned}s_m[z + A^{1/2}] &= s_m(z)\exp[\mathrm{i}z_{\mathrm{I}}A^{1/2}],\\ s_m[z + \mathrm{i}A^{1/2}] &= s_m(z)\exp[-\mathrm{i}z_{\mathrm{R}}A^{1/2}].\end{aligned} \tag{436}$$

Using them we can prove a resolution of the identity in $\mathcal{H}_{2j+1}$ that involves the coherent states $|z\rangle_{\mathrm{cf}}$ in a square cell with area A:

$$S = [a, a + A^{1/2})_{\mathrm{R}} \times [b, b + A^{1/2})_{\mathrm{I}}. \tag{437}$$

Here a, b are arbitrary real numbers and the cell can be shifted everywhere in the complex plane. The resolution of the identity is given by

$$A^{-1/2} \int_{\mathrm{S}} d^2 z\, \mathcal{N}(z) |z\rangle_{\mathrm{cf}\ \mathrm{cf}}\langle z| = \mathbf{1}. \tag{438}$$

18.2. Theta function representation on a torus

We consider an arbitrary state $|f\rangle$ in $\mathcal{H}_{2j+1}$:

$$|f\rangle = \sum_m f_m |j; m\rangle_J. \tag{439}$$

We represent this state with the following analytic function [179]:

$$\begin{aligned} f_{\mathrm{T}}(z) &\equiv [\mathcal{N}(z)]^{1/2} \pi^{-1/2} A^{1/2} \exp(-\mathrm{i} z_{\mathrm{I}} z)\ {}_{\mathrm{cf}}\langle z^* | f\rangle \\ &= \pi^{-1/4} \sum_{m=0}^{2j} f_m \Theta_3 \left[-\pi^2 m A^{-1} + z\pi A^{-1/2}; \frac{\mathrm{i}}{2j+1} \right], \end{aligned} \tag{440}$$

where the index ‘T’ stands for ‘theta function representation’.

The function $f_{\mathrm{T}}(z)$ is quasi-periodic

$$\begin{aligned} f_{\mathrm{T}}[z + A^{1/2}] &= f_{\mathrm{T}}(z), \\ f_{\mathrm{T}}[z + \mathrm{i}A^{1/2}] &= f_{\mathrm{T}}(z) \exp[A - 2\mathrm{i}A^{1/2} z]. \end{aligned} \tag{441}$$

For this reason, we only consider it in the cell S defined in equation (437); in other words, $f_{\mathrm{T}}(z)$ is defined on a torus. However, we stress the fact that it is not an elliptic (doubly-periodic) function. A (non-constant) elliptic function has poles. The function $f_{\mathrm{T}}(z)$ is quasi-periodic along the imaginary axis and has no poles. If we consider $f_{\mathrm{T}}(z)$ in the complex plane which is the covering surface of the torus, we easily see that it has growth with order $\rho = 2$.

Another related analytic representation is to represent the state $|f\rangle$ with the function

$$\begin{aligned} F_{\mathrm{T}}(z) &= \sum_{m=0}^{2j} f_m \Theta_3[-\mathrm{i}\pi m + \mathrm{i}z A^{1/2}; \mathrm{i}(2j+1)] \exp[2\pi z m A^{-1/2} - \pi^2 m^2 A^{-1}] \\ &= \pi^{3/4} A^{-1/2} \exp(z^2) f_{\mathrm{T}}(z). \end{aligned} \tag{442}$$

The function $F_{\mathrm{T}}(z)$ is also quasi-periodic

$$\begin{aligned} F_{\mathrm{T}}[z + A^{1/2}] &= F_{\mathrm{T}}(z) \exp(A + 2A^{1/2} z), \\ F_{\mathrm{T}}[z + \mathrm{i}A^{1/2}] &= F_{\mathrm{T}}(z). \end{aligned} \tag{443}$$

Using the resolution of the identity of equation (438), we find that the scalar product is given by

$$\begin{aligned} \langle f^* | g\rangle &= (\pi)^{-1/2} (2j+1)^{-3/2} \int_{\mathrm{S}} \mathrm{d}^2 z \exp\left(-2z_{\mathrm{I}}^2\right) f_{\mathrm{T}}(z) g_{\mathrm{T}}(z^*) \\ &= \pi (2j+1)^{-1/2} \int_{\mathrm{S}} \mathrm{d}^2 z \exp\left(-2z_{\mathrm{R}}^2\right) F_{\mathrm{T}}(z) G_{\mathrm{T}}(z^*). \end{aligned} \tag{444}$$

As examples, we mention that the angular momentum state $|j; m\rangle_J$ is represented with the function

$$f_{\mathrm{T}}(z) = \pi^{-1/4} \Theta_3 \left[-\pi^2 m A^{-1} + z\pi A^{-1/2}; \frac{\mathrm{i}}{2j+1} \right] \tag{445}$$

and the angle state $|j; m\rangle_\theta$ with the function

$$f_{\mathrm{T}}(z) = \pi^{-1/4}\exp(-z^2)\Theta_3\left[-\pi^2 m A^{-1} + \mathrm{i}z\pi A^{-1/2}; \frac{\mathrm{i}}{2j+1}\right]. \tag{446}$$

We next discuss transformations and two important classes are the displacements and the symplectic transformations. We first consider the displacements of equation (373) in the θ_z–J_z angle–angular momentum phase space which is the toroidal lattice $\mathcal{Z}_{2j+1} \times \mathcal{Z}_{2j+1}$. The operators E and G are implemented in the theta representation as follows:

$$\begin{aligned} E &= \exp[-\pi A^{-1/2}\partial_z], \\ G &= \exp[\mathrm{i}2\pi A^{-1/2}z - \pi^2 A^{-1}]\exp[\mathrm{i}\pi A^{-1/2}\partial_z]. \end{aligned} \tag{447}$$

We can check that these operators obey the Heisenberg–Weyl group relations of equation (375), and also that acting with them on the angular momentum states of equation (445) and the angle states of equation (446) we get the relations of equation (374). We note that the relation $E^{2j+1} = \mathbf{1}$ is related to the periodicity of the function $f_{\mathrm{T}}(z)$ along the real axis. Also the relation $G^{2j+1} = \mathbf{1}$ is related to the quasi-periodicity of the function $f_{\mathrm{T}}(z)$ along the imaginary axis.

Symplectic transformations in the toroidal lattice $\mathcal{Z}_{2j+1} \times \mathcal{Z}_{2j+1}$ have been reviewed in [69]. They are well defined in the case that $2j+1 = p^m$ where p is a prime number. In this case, $\mathcal{Z}_{2j+1}$ becomes the Galois field $GF(p^m)$ and the lattice phase space is a finite geometry where translations and rotations are well defined and they form groups. Further work is needed to study these transformations in the language of theta functions, using the theta function representation. Symplectic transformations in the context of theta functions is a well studied subject and Mumford [178] summarized this work using the more general theta functions

$$\Theta_{\mu,\nu}(u;\tau) = \sum_{n=-\infty}^{\infty}\exp\left[\mathrm{i}\pi\tau\left(n+\frac{\mu}{2}\right)^2 + 2\mathrm{i}\pi u\left(n+\frac{\mu}{2}\right) + \mathrm{i}\pi n\nu\right]. \tag{448}$$

The link between symplectic transformations in the context of quantum systems with dimension which is the power of a prime, and the mathematical work on symplectic transformations for theta functions, requires further study.

18.3. Zeros of the functions $f_{\mathrm{T}}(z)$

As with the other analytic representations, the zeros of $f_{\mathrm{T}}(z)$ correspond to coherent states which are orthogonal to the state $|f\rangle$ (equation (440)).

We consider the function $f_{\mathrm{T}}(z)$ in the complex plane (which is the covering surface of the torus) and using the criterion of equation (93) and the quasi-periodicity of equation (441) we prove that

$$\oint_\ell \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\partial_z f_{\mathrm{T}}(z)}{f_{\mathrm{T}}(z)} = 2j+1, \tag{449}$$

where ℓ is a contour along the boundary of the cell S. This shows that the function $f_{\mathrm{T}}(z)$ has exactly $2j+1$ zeros ζ_n (with the multiplicities taken into account) within the cell S which has area $\pi(2j+1)$. Using equation (92) and the quasi-periodicity of equation (441), we show that the 'centre of mass' of the zeros satisfies the following constraint:

$$\oint_\ell \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\partial_z f_{\mathrm{T}}(z)}{f_{\mathrm{T}}(z)}z = \sum_{n=1}^{2j+1}\zeta_n = A^{1/2}\left[M + \mathrm{i}N + (1+\mathrm{i})\left(j+\frac{1}{2}\right)\right], \tag{450}$$

where M, N are integers and A is the area of the square cell. Equations (449) and (450) have been discussed in [42, 179].

We next assume that the $2j + 1$ zeros ζ_n in the cell S are given, and that they obey the constraint of equation (450). We will construct the function $f_{\mathrm{T}}(z)$ [179]. We first consider the product

$$f_1(z) = \prod_{n=1}^{2j+1} \Theta_3[w_n(z); \mathrm{i}], \qquad w_n(z) = \pi A^{-1/2}(z - \zeta_n) + \frac{\pi(1 + \mathrm{i})}{2}, \quad (451)$$

which has the given zeros. The ratio $f_{\mathrm{T}}(z)/f_1(z)$ is an entire function with no zeros and therefore it is the exponential of an entire function. Taking into account the periodicity constraints of equation (441) and the fact that that the growth of $f_{\mathrm{T}}(z)$ is of order $\rho = 2$, we conclude that

$$f_{\mathrm{T}}(z) = C \exp[-\mathrm{i}2\pi A^{-1/2} N z] \prod_{n=1}^{2j+1} \Theta_3[w_n(z); \mathrm{i}], \quad (452)$$

where N is the integer in the constraint of equation (450) and C is a constant determined by the normalization condition. We can factorize further the function $f_{\mathrm{T}}(z)$ using Jacobi's triple product identity [182]. For $|x| < 1$ and $z \neq 0$,

$$\prod_{m=1}^{\infty} (1 - x^{2m})(1 + x^{2m-1} z^2)(1 + x^{2m-1} z^{-2}) = \sum_{n=-\infty}^{\infty} x^{n^2} z^{2n}. \quad (453)$$

Using this we express the function $f_{\mathrm{T}}(z)$ as

$$\begin{aligned} f_{\mathrm{T}}(z) = C \exp[-\mathrm{i}2\pi A^{-1/2} N z] \prod_{m=1}^{\infty} \Bigg[&(1 - \mathrm{e}^{-2\pi m})^{2j+1} \\ &\times \prod_{n=1}^{2j+1} (1 + \mathrm{e}^{-\pi(2m-1)+2\mathrm{i}w_n(z)})(1 + \mathrm{e}^{-\pi(2m-1)-2\mathrm{i}w_n(z)}) \Bigg]. \end{aligned} \quad (454)$$

19. Discussion

In this review, we have reviewed the work on analytic representations in quantum mechanics. We have discussed Euclidean, hyperbolic and elliptic analytic representations. The basic transformations are $D(z)$, $S(z; k)$ and $\mathcal{R}(z; j)$ of equations (32), (172) and (339) in these three cases, and they obey the generalized resolutions of the identity given in equations (38), (176) and (343).

In the Euclidean case, the Bargmann analytic representation in the complex plane is intimately connected with Glauber coherent states and more generally with the phase-space formalism of the harmonic oscillator. We have introduced briefly displacement operators and Weyl functions, displaced parity operators and Wigner functions, coherent states and P and Q functions, and we have explained how they are related to the Bargmann functions. For example, in equations (110)–(113) we have given the relation between the Bargmann kernel of an operator and its P, Q and Wigner and Weyl functions. We have also discussed the Euclidean Berezin formalism. In equations (119), we have shown that in the semiclassical limit the $\mathcal{L}_{\mathrm{E}}$-kernel of the commutator of two operators becomes the Poisson bracket of the two $\mathcal{L}_{\mathrm{E}}$-kernels of these operators.

The general theory that relates the growth with the density of zeros of analytic functions has been applied to Bargmann functions. Physically, the zeros of a Bargmann function representing a particular state, correspond to Glauber coherent states which are orthogonal to this state. Consequently, many mathematical theorems on the density of zeros of analytic functions acquire physical meaning. This leads to results on the completeness of sequences of Glauber coherent states. In the case of a sequence of Glauber coherent states which is undercomplete, there are states which are orthogonal to all coherent states in the sequence, and Hadamard's theorem explains how to construct them. From a more applied point of view, as a state evolves in time its zeros also evolve with time as we have seen in a very simple example in equations (141) and (143). In more complicated systems, the motion of zeros can provide a deeper insight about the system and this has been used in quantum maps and quantum chaos.

The Euclidean contour representation is another analytic representation based on contour integrals. We have explained that when the ket function $f_{\mathrm{Ek}}(z)$ has order of the growth $1 < \rho \leqslant 2$ there might be convergence difficulties with the corresponding function $f_{\mathrm{Eb}}(z)$. Notwithstanding this weakness, the method has been used in the literature [45–51].

In the hyperbolic case, we have discussed several analytic representations. We first discussed $SU(1,1)$ coherent states and the Barut–Girardello states. We also discussed implementations of some $SU(1,1)$ representations in the harmonic oscillator context. For example, the $k = 1/4$ and $k = 3/4$ representations have been used in connection with the subspaces $\mathcal{H}_{\mathrm{even}}$ and $\mathcal{H}_{\mathrm{odd}}$ of the harmonic oscillator. This leads to parity-dependent squeezing in equation (220). Another example is the $k = 1/2$ representation which is used for the description of phase states of the harmonic oscillator. Another example is the Schwinger representation of $SU(1,1)$ which is used in quantum optics for the description of amplifiers. Generalizations of this formalism to larger groups have many applications in various branches of physics.

$SU(1,1)$ coherent states have been used to define analytic representations in the unit disc, which belong in the Bergman space. In this representation, $SU(1,1)$ transformations are implemented with Möbius conformal mappings as described in equation (249). We have also discussed the hyperbolic Berezin formalism and we have shown in equation (268) that in the $k \to \infty$ limit the $\mathcal{L}_{\mathrm{H}}$-kernel of the commutator of two operators becomes the hyperbolic Poisson bracket of the two $\mathcal{L}_{\mathrm{E}}$-kernels of these operators.

Another hyperbolic analytic representation in the unit disc is based on $SU(1,1)$ phase states and uses functions in the Hardy space. Such function is factorized in terms of its outer part which is related to the phase distribution of the corresponding quantum state, and its inner part which contains all the zeros. Theorems like equation (298) about the zeros are interpreted as criteria about the overcompleteness of the phase states.

The Barut–Girardello analytic representation is defined in the complex plane. Equations (313) and (314) relate it to the hyperbolic analytic representation in the unit disc based on $SU(1,1)$ coherent states. It can be used in the context of the $k = 1/4$ and $k = 3/4$ representations in connection with the subspaces $\mathcal{H}_{\mathrm{even}}$ and $\mathcal{H}_{\mathrm{odd}}$ of the harmonic oscillator. In this case, there is a relationship between the Barut–Girardello analytic representation and the Bargmann representation which is given in equation (320).

The hyperbolic contour representation is based on the functions defined in equation (322). The ket function converges at least in the interior of the unit disc, and the bra function converges at least in the exterior of the unit disc. The formalism requires that both of these functions are defined within an annulus and we discussed the conditions for this.

Elliptic analytic representations are based on $SU(2)$ coherent states. We have discussed $SU(2)$ coherent states, the Holstein–Primakoff $SU(2)$ formalism and the Schwinger $SU(2)$ formalism (which has been used extensively in quantum optics for the description of frequency converters). The elliptic analytic representation in the extended complex plane has been defined in equation (390). In this representation, $SU(2)$ transformations are implemented with Möbius conformal mappings as described in equation (398). We have also discussed the elliptic Berezin formalism and we have shown in equation (417) that in the $j \to \infty$ limit the $\mathcal{L}_S$-kernel of the commutator of two operators becomes the elliptic Poisson bracket of the two $\mathcal{L}_S$-kernels of these operators.

The elliptic contour representation has been defined in equation (418). The ket function $f_{Sk}(z; j)$ has singularity at ∞ and the bra function $f_{Sb}(z; j)$ has singularity at 0. The convergence difficulties of the corresponding formalism in the Euclidean and hyperbolic cases do not appear here.

An analytic representation of finite quantum systems on a torus which is based on theta functions, has been defined in equation (440). The function $f_T(z)$ obeys the quasi-periodicity relations of equation (441) and therefore it is sufficient to define it in the square cell S of equation (437) which has area $\pi(2j+1)$. It has been shown that within the cell S the function $f_T(z)$ has exactly $2j+1$ zeros. Equation (452) shows how to construct the function $f_T(z)$ from its zeros.

We next discuss the merits of analytic representations in comparison to the non-analytic ones. Some results can only be derived in the language of analytic representations. Examples are the criteria for completeness of sequences of coherent states, the expansion of arbitrary states in terms of coherent states on a contour given in equations (165), (332), (427), etc. Other problems could be tackled in any representation, but it might be easier to work with analytic representations. For example, the use of the conformal mappings of equations (249) and (398) might be easier than working with the $SU(1, 1)$ and $SU(2)$ algebras in non-analytic representations. In this sense, these mappings might be good practical tools in calculations of the time evolution of certain systems.

In conclusion, we indicate some areas for further work. There is already a lot of work in generalizing $SU(2)$ coherent states to $SU(N)$ coherent states [183]. This work could lead to more general analytic representations and further work is required in this direction.

We have discussed analytic representations in the complex plane, unit disc and extended complex plane (sphere). The next step is analytic representations in multiply connected Riemann surfaces and the simplest case is the torus (which has genus $g = 1$ and the complex plane as covering surface). We have discussed a particular representation of finite quantum systems on a torus using theta functions in section 18, and references [39, 41–44] have discussed related work in the context quantum maps, but more work could be done in this direction. The next step is analytic representations in Riemann surfaces with genus $g \geqslant 2$ (with the unit disc as covering surface). They have been studied extensively in the context of conformal field theories [94, 95], but there is very little work in a context relevant to this review. Further work is required to explore their use in the areas of coherent states, quantum optics, quantum information processing, condensed matter, etc.

The representation of multimode systems with analytic functions of many variables, and the possible use of analyticity in the study of timely problems like entanglement [184], is also another problem for further work.

Analytic representations have already been used extensively in the general area of mathematical physics. Perhaps, they ought to be utilized more widely in more applied contexts which use coherent states and we hope that this review will create interest in this direction.

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